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Algebra II Study Guide

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Eureka Math Algebra II Study Guide

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Contents

Introduction

When do you know you really understand something? One test is to see if you can explain it to someone else—well enough that *they* understand it. *Eureka Math* routinely requires students to "turn and talk" and explain the math they learned to their peers.

That is because the goal of *Eureka Math* (which you may know as the EngageNY math modules) is to produce students who are not merely literate, but fluent, in mathematics. By fluent, we mean not just knowing what process to use when solving a problem but understanding why that process works.

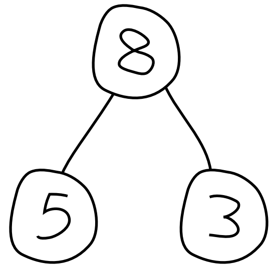
Here's an example. A student who is fluent in mathematics can do far more than just name, recite, and apply the Pythagorean theorem to problems. She can explain why $a^2 + b^2 = c^2$ is true. She not only knows that the theorem can be used to find the length of a right triangle's hypotenuse but also can apply it more broadly—such as to find the distance between any two points in the coordinate plane, for example. She also can see the theorem as the glue joining seemingly disparate ideas including equations of circles, trigonometry, and vectors.

By contrast, the student who has merely memorized the Pythagorean theorem does not know why it works and can do little more than just solve right triangle problems by rote. The theorem is an abstraction—not a piece of knowledge, but just a process to use in the limited ways that she has been directed. For her, studying mathematics is a chore, a mere memorizing of disconnected processes.

Eureka Math provides much more. It offers students math knowledge that will serve them well beyond any test. This fundamental knowledge not only makes wise citizens and competent consumers but also gives birth to budding physicists and engineers. Knowing math deeply opens vistas of opportunity.

Students become fluent in math—as they do in any other subject—by following a course of study that builds their knowledge of the subject, logically and thoroughly. In *Eureka Math*, concepts flow logically from PreKindergarten through high school. The "chapters" in the story of mathematics are *A Story of Units* for the elementary grades, followed by *A Story of Ratios* in middle school, and *A Story of Functions* in high school.

This sequencing is joined with a mix of new and old methods of instruction that are proven to work. For example, we utilize an exercise called a "sprint" to develop students' fluency with standard algorithms (routines for adding, subtracting, multiplying, and dividing whole numbers and fractions). We employ many familiar models and tools such as the number line and tape diagrams (aka bar models). A newer model highlighted in the curriculum is the number bond (illustrated on the following page), which clearly shows how numbers are composed of other numbers.



Eureka Math is designed to help accommodate different types of classrooms and to serve as a resource for educators, who make decisions based on the needs of students. The "vignettes" of teacher-student interactions included in the curriculum are not scripts, but exemplars illustrating methods of instruction recommended by the teachers who have crafted our curriculum.

Eureka Math has been adopted by districts from East Meadows, New York, to Lafayette, Louisiana, to Chula Vista, California. At *Eureka Math* we are excited to have created the most transparent math curriculum in history—every lesson, all classwork, and every problem is available online.

Many of us have less than joyful memories of learning mathematics: lots of memorization, lots of rules to follow without understanding, and problems that didn't make any sense. What if a curriculum came along that gave children a chance to avoid that math anxiety and replaced it with authentic understanding, excitement, and curiosity? Like a New York educator attending one of our trainings said: "Why didn't I learn mathematics this way when I was a kid? It is so much easier than the way I learned it!"

Eureka!

Lynne Munson
Washington, DC

From the Writer

To My Fellow Teachers:

I was hooked into mathematics by the intricacy and beauty of high school geometry. I distinctly remember being home sick from school and constructing a new-to-me proof of the Pythagorean theorem for the sheer joy of solving the puzzle. My interest in mathematics grew through subsequent courses in algebra, trigonometry, and calculus, although I didn't fully grasp the connections among these fields of mathematics until I began to teach.

Classroom teachers and mathematicians from across the country have come together to develop the *Eureka Math* curriculum to bring these connections to life. For example, in Algebra II students will see that we can use the analytic concept of the average rate of change to lead to the geometric formulas for the area of a circle and the volume of a sphere. They will see that we can apply geometric concepts of congruence and similarity to parabolas in the coordinate plane and that the study of motion around a circle—the simplest geometric object—leads to an entire field of mathematics that encompasses and expands on the triangle trigonometry studied in Geometry.

This course that we share with you is full of deep and beautiful mathematics, carefully sequenced to coherently build on the foundations of Algebra I and Geometry previously constructed. We have developed and refined the modules with great care; however, we envision that you will adapt these materials to meet the needs of your particular students, collaborating with your colleagues to tinker with the lessons and make them your own.

Speaking for the teachers and mathematicians who outlined, wrote, critiqued, rewrote, edited, and revised these lessons, we thank you for the careful deliberation you put into your preparation every day in order to promote student success. We consider you to be a crucial part of our team, and we look forward to your input on how we can continue to improve the Algebra II curriculum.

Chris Black
Seattle, WA
Algebra II lead writer/editor
Eureka Math/Great Minds

Foreword

TELLING THE STORY OF MATH

Each module in *Eureka Math* builds carefully and precisely on the content learned in the previous modules and years, weaving the knowledge learned into a coherent whole. This produces an effect similar to reading a good novel: The storyline, even after weeks of not reading, is easy to pick up again because the novel pulls the reader back into the plot immediately—the need to review is minimal because the plot brings out and adds to what has already happened. This cumulative aspect of the plot, along with its themes, character development, and composition, are all part of the carefully thought-out design of the *Eureka Math* curriculum.

So what is the storyline? One can get a sense of how the story evolves by studying the major themes of *A Story of Units*, *A Story of Ratios*, and *A Story of Functions*.

A Story of Units investigates how concepts including place value, algorithms, fractions, measurements, area, and so on can all be understood by relating and manipulating types of units (e.g., inches, square meters, tens, fifths). For example, quantities expressed in the same units can be added: 3 apples plus 4 apples equals 7 apples. Likewise, 3 fifths plus 4 fifths is 7 fifths. Whole number multiplication, as in "3 fives = 15 ones," is merely another form of converting between different units, as when we state that "1 foot = 12 inches." These similarities between concepts drive the day-to-day theme throughout the PreK–5 curriculum: each type of unit (or building block) is handled the same way through the common features that all units share. Understanding the commonalities and like traits of these building blocks makes it much easier to sharply contrast the differences. In other words, the consistency of manipulation of different units helps students see the connection in topics. No longer is every new topic separate from the previous topics studied.

A Story of Ratios moves students beyond problems that involve one-time calculations using one or two specific measurements to thinking about proportional relationships that hold for a whole range of measurements. The proportional relationships theme shows up every day during middle school as students work with ratios, rates, percentages, probability, similarity, and linear functions. *A Story of Ratios* provides the transition years between students thinking of a specific triangle with side lengths 3 cm, 4 cm, and 5 cm in elementary school to a broader view in high school for studying the set of all triangles with side lengths in a 3:4:5 ratio (e.g., 6:8:10, 9:12:15).

A Story of Functions generalizes linear relationships learned in middle school to polynomial, rational, trigonometric, exponential, and logarithmic functions in high school. Students study the properties of these functions and their graphs and model with them to move explicitly from real-world scenarios to mathematical representations. The algebra learned in middle school is applied in rewriting functions in different forms and solving equations derived from one or more functions. The theme drives students to finish high

school knowing not only how to manipulate the major functions used in college but also how to be fully capable of modeling real-life data with an appropriate function in order to make predictions and answer questions.

The many “little eurekas” infused in the storyline of *Eureka Math* help students learn how to wield the true power of mathematics in their daily lives. Experiencing these “aha moments” also convinces students that the mathematics that drives innovation and advancement in our society is within their reach.

Scott Baldridge
Lead writer and lead mathematician, *Eureka Math*
Loretta Cox Stuckey and Dr. James G. Traynham Distinguished Professor of Mathematics, Louisiana State University
Co-director, Gordon A. Cain Center for Science, Technology, Engineering, and Mathematical Literacy

How to Use This Book

As a self-study resource, these *Eureka Math* Study Guides are beneficial for teachers in a variety of situations. They introduce teachers who are brand new to either the classroom or the *Eureka Math* curriculum not only to *Eureka Math* but also to the content of the grade level in a way they will find manageable and useful. Teachers already familiar with the curriculum will also find this resource valuable as it allows a meaningful study of the grade-level content in a way that highlights the connections between modules and topics. The guidebooks help teachers obtain a firm grasp on what it is that students should master during the year. The structure of the book provides a focus on the connections between the standards and the descriptions of mathematical progressions through the grade, topic by topic. Teachers therefore develop a multifaceted view of the standards from a thorough analysis of the guide.

The *Eureka Math* Study Guides can also serve as a means to familiarize teachers with adjacent grade levels. It is helpful for teachers to know what students learned in the grade level below the one they are currently teaching as well as the one that follows. Having an understanding of the mathematical progression across grades enhances the teacher's ability to reach students at their level and ensure they are prepared for the next grade.

For teachers, schools, and districts that have not adopted *Eureka Math*, but are instead creating or adjusting their own curricular frameworks, these grade-level study guides offer support in making critical decisions about how to group and sequence the standards for maximal coherence within and across grades. *Eureka Math* serves as a blueprint for these educators; in turn, the study guides present not only this blueprint but a rationale for the selected organization.

The *Eureka Math* model provides a starting point from which educators can build their own curricular plan if they so choose. Unpacking the new standards to determine what skills students should master at each grade level is a necessary exercise to ensure appropriate choices are made during curriculum development. The *Eureka Math* Study Guides include lists of student outcomes mapped to the standards and are key to the unpacking process. The overviews of the modules and topics offer narratives rich with detailed descriptions of how to teach specific skills needed at each grade level. Users can have confidence in the interpretations of the standards presented, as well as the sequencing selected, due to the rigorous review process that occurred during the development of the content included in *Eureka Math*.

This *Eureka Math* Study Guide contains the following:

Introduction to Eureka Math (chapter 1): This introduction consists of two sections: "Vision and Storyline" and "Advantages to a Coherent Curriculum."

Major Mathematical Themes in Each Grade Band (chapter 2): The first section presents year-long curriculum maps for each grade band (with subsections addressing *A Story of Units*, *A Story of Ratios*, and *A Story of Functions*). It is followed by a detailed examination of math concept development for courses typically taught from Grade 9 to Grade 12. The chapter closes with an in-depth description of how alignment to the Instructional Shifts and the Standards of Mathematical Practice is achieved.

Course Content Review (chapter 3): The purpose and recommended fluencies for the course are presented in this chapter, along with a rationale for why topics are grouped and sequenced in the modules as they are. The Alignment to the Standards and Placement of Standards in the Modules chart lists the standards that are addressed in each module of the course.

Curriculum Design (chapter 4): The approach to modules, lessons, and assessment in *A Story of Functions* is detailed in this chapter.

Approach to Differentiated Instruction (chapter 5): This chapter describes the approach to differentiated instruction used in *A Story of Functions*. Special populations such as English language learners, students with disabilities, students performing above grade level, and students performing below grade level are addressed.

Course Module Summary and Unpacking of Standards (chapter 6): This chapter presents information from the modules to provide an overview of the content of each and explain the mathematical progression. The standards are translated for teachers, and a fuller picture is drawn of the teaching and learning that should take place through the school year.

Terminology (chapter 7): The terms included in this list were compiled from the New or Recently Introduced Terms portion of the Terminology section of the Module Overviews. Terms are listed by course and module number where they are introduced in *A Story of Functions*, and definitions for these terms are provided.

Eureka Math Algebra II Study Guide

CHAPTER 1

Introduction to Eureka Math

VISION AND STORYLINE

Eureka Math is a comprehensive, content-rich PreK–12 curriculum and professional development platform. It follows the focus and coherence of the new college- and career-ready standards and carefully sequences the mathematical progressions into expertly crafted instructional modules.

The new standards and progressions set the frame for the curriculum. We then shaped every aspect of it by addressing the new Instructional Shifts that teachers must make. Nowhere are the Instructional Shifts more evident than in the fluency, application, concept development, and debriefing sections that characterize lessons in the PreK–5 grades of *Eureka Math.* Similarly, Eureka's focus in the middle and high school grades on problem sets, exploration, Socratic discussion, and modeling helps students internalize the true meaning of coherence and fosters deep conceptual understanding.

Eureka Math is distinguished not only by its adherence to the new standards, but also by its foundation in a theory of teaching math that has been proven to work. This theory posits that mathematical knowledge is conveyed most effectively when it is taught in a sequence that follows the story of mathematics itself. This is why we call the elementary portion *A Story of Units*, followed by *A Story of Ratios* in middle school, and *A Story of Functions* in high school. Mathematical concepts flow logically from one to the next in this curriculum.

The sequencing has been joined with proven methods of instruction. These methods drive student understanding beyond process to deep mastery of mathematical concepts. The goal of *Eureka Math* is to produce students who are fluent, not merely literate, in mathematics.

In spite of the extensiveness of these resources, *Eureka Math* is not meant to be prescriptive. Rather, we offer it as a basis for teachers to hone their own craft. Great Minds believes deeply in the ability of teachers and in their central, irreplaceable role in shaping the classroom experience. To support and facilitate that important work, *Eureka Math* includes

both scaffolding hints to help teachers support Response to Intervention (RTI) and maintains a consistent lesson structure that allows teachers to focus their energy on engaging students in the mathematical story.

In addition, the online version of *Eureka Math* (www.eureka-math.org) features embedded videos that demonstrate classroom practices. The readily navigable online version includes progressions-based search functionality to permit navigation between standards and related lessons, linking all lessons in a particular standard's strand or mathematical progression and learning trajectory. This functionality also helps teachers identify and remediate gaps in prerequisite knowledge, implement RTI tiers, and provide support for students at a variety of levels.

The research and development on which *Eureka Math* is based was made possible through a partnership with the New York State Education Department, for which this work was originally created. The department's expert review team, including renowned mathematicians who helped write the new standards, progressions, and the much-touted "Publishers' Criteria" (http://achievethecore.org/page/686/publishers-criteria), strengthened an already rigorous development process. We are proud to offer *Eureka Math*, an extended version of that work, to teachers all across the country.

ADVANTAGES TO A COHERENT CURRICULUM

Great Minds believes in the theory of teaching content as a coherent story from PreK to Grade 12—one that is sequential, scaffolded, and logically cohesive within and between grades. Great Minds' *Eureka Math* is a program with a three-part narrative, from *A Story of Units* (PreK–5) to *A Story of Ratios* (6–8) to *A Story of Functions* (9–12). This curriculum shows Great Minds' commitment to provide educators with the tools necessary to move students between grade levels so that their learning grows from what comes before and after.

A coherent curriculum creates a common knowledge base for all students that supports effective instruction across the classroom. Students' sharing of a base of knowledge engenders a classroom environment of common understanding and learning. This means that the effectiveness of instruction can be far more significant than when topics are taught as discrete unrelated items, because teachers can work with students to achieve a deep level of comprehension and shared learning.

This cohesiveness must be based on the foundation of a content-rich curriculum that is well organized and thoughtfully designed in order to facilitate learning at the deepest level. A coherent curriculum should be free of gaps and needless repetition, aligned to standards but also vertically and horizontally linked across lessons and grade levels. What students learn in one lesson prepares them for the next in a logical sequence. In addition, what happens in one second-grade classroom in one school closely matches what happens in another second-grade classroom, creating a shared base of understanding across students, grades, and schools.

Lack of coherence can lead to misalignment and random, disordered instruction that can prove costly to student learning and greatly increase the time that teachers spend on preparation, revisions, and repetition of material. The model of a sequential, comprehensive

curriculum, such as *Eureka Math*, brings benefits within the uniformity in time spent on content, approach to instruction, and lesson structure, facilitating a common base of knowledge and an environment of shared understanding.

The commitment to uniformity influenced Great Minds' approach to creating *Eureka Math*. This curriculum was created from a single vision spanning PreK–12, with the same leadership team of mathematicians, writers, and project managers overseeing and coordinating the development of all grades at one time. By using the same project team throughout the course of *Eureka Math*'s development, Great Minds was able to ensure that *Eureka Math* tells a comprehensive story with no gaps from grade to grade or band to band.

CHAPTER 2

Major Mathematical Themes in Each Grade Band

This chapter presents the year-long curriculum maps for each grade band in the *Eureka Math* curriculum: A *Story of Units*, A *Story of Ratios*, and A *Story of Functions*. These maps illustrate the major mathematical themes across the entire mathematics curriculum. The chapter also includes a detailed examination of the math concept development for A *Story of Functions*. The chapter closes with an in-depth description of how the curriculum is aligned to the Instructional Shifts and the Standards for Mathematical Practice.

YEAR-LONG CURRICULUM MAPS FOR EACH GRADE BAND

The curriculum map is a chart that shows, at a glance, the sequence of modules that make up each grade or course of the entire curriculum for a given grade band. The map also indicates the approximate number of instructional days designated for each module of each grade or course. It is important for educators to have knowledge of how key topics are sequenced from PreK through Grade 12. The maps for the three grade bands in Figures 2.1 to 2.3 reveal the trajectories through the grades for topics such as geometry, fractions, functions, and statistics and probability.

MATH CONTENT DEVELOPMENT FOR GRADES 9–12: A STORY OF FUNCTIONS

The curricular design for A *Story of Functions* is based on the principle that mathematics is most effectively taught as a logical, engaging story. At the high school level, the story's main plot line is the study of problems in science, technology, engineering, and mathematics that are modeled by functions. Through these problems, students are introduced to and learn the properties of the main types of functions. For example, students model bacteria growth using

	Pre-Kindergarten	Kindergarten	Grade 1	Grade 2
1st TRIMESTER	M1: Counting to 5 (45 days)	M1: Numbers to 10 (45 days)	M1: Sums and Differences to 10 (45 days)	M1: Sums and Differences to 100 (10 days)
	M2: Shapes (15 days)	**M2: 2D and 3D Shapes (12 days)	M2: Introduction to Place Value Through Addition and Subtraction Within 20 (35 days)	M2: Addition and Subtraction of Length Units (12 days)
2nd TRIMESTER	M3: Counting to 10 (50 days)	M3: Comparison of Length, Weight, Capacity, and Numbers to 10 (38 days)	M3: Ordering and Comparing Length Measurements as Numbers (15 days)	M3: Place Value, Counting, and Comparison of Numbers to 1,000 (25 days)
	M4: Comparison of Length, Weight, Capacity, and Numbers to 5 (35 days)	M4: Number Pairs, Addition and Subtraction to 10 (47 days)	M4: Place Value, Comparison, Addition and Subtraction to 40 (35 days)	M4: Addition and Subtraction Within 200 with Word Problems to 100 (35 days)
3rd TRIMESTER	M5: Addition and Subtraction Stories and Counting to 20 (35 days)	M5: Numbers 10–20 and Counting to 100 (30 days)	M5: Identifying, Composing, and Partitioning Shapes (15 days)	M5: Addition and Subtraction Within 1,000 with Word Problems to 100 (24 days)
		M6: Analyzing, Comparing, and Composing Shapes (10 days)	M6: Place Value, Comparison, Addition and Subtraction to 100 (35 days)	M6: Foundations of Multiplication and Division (24 days)
				M7: Problem Solving with Length, Money, and Data (30 days)
				M8: Time, Shapes, and Fractions as Equal Parts of Shapes (20 days)

Key:	Number	Geometry

*The columns indicating trimesters and quarters are provided to give you a rough guideline. Please use your own pacing considerations based on the specific dates of your academic calender.
**Please refer to the modules themselves to identify partially labeled titles as well as the standards corresponding to all modules.

Figure 2.1 Grades PreK–5 Year-Long Curriculum Map: A Story of Units

Grade 3	Grade 4	Grade 5
M1: Properties of Multiplication and Division and Solving Problems with Units of 2–5 and 10 (25 days)	M1: Place Value, Rounding, and Algorithms for Addition and Subtraction (25 days)	M1: Place Value and Decimal Fractions (20 days)
M2: Place Value and Problem Solving with Units of Measure (25 days)	**M2: Unit Conversions (7 days)	M2: Multi-Digit Whole Number and Decimal Fraction Operations (35 days)
M3: Multiplication and Division with Units of 0, 1, 6–9, and Multiples of 10 (25 days)	M3: Multi-Digit Multiplication and Division (43 days)	M3: Addition and Subtraction of Fractions (22 days)
M4: Multiplication and Area (20 days)	M4: Angle Measure and Plane Figures (20 days)	M4: Multiplication and Division of Fractions and Decimal Fractions (38 days)
M5: Fractions as Numbers on the Number Line (35 days)	M5: Fraction Equivalence, Ordering, and Operations (45 days)	M5: Addition and Multiplication with Volume and Area (25 days)
M6: Collecting and Displaying Data (10 days)	M6: Decimal Fractions (20 days)	M6: Problem Solving with the Coordinate Plane (40 days)
M7: Geometry and Measurement Word Problems (40 days)	M7: Exploring Measurement with Multiplication (20 days)	

1st QUARTER

2nd QUARTER

3rd QUARTER

4th QUARTER

Number and Geometry, Measurement	Fractions

	Grade 6	Grade 7	Grade 8	
1st TRIMESTER	M1: Ratios and Unit Rates (35 days)	M1: Ratios and Proportional Relationships (30 days)	M1: Integer Exponents and Scientific Notation (20 days)	1st QUARTER
	M2: Arithmetic Operations Including Division of Fractions (25 days)	M2: Rational Numbers (30 days)	M2: The Concept of Congruence (25 days)	
2nd TRIMESTER	M3 Rational Numbers (25 days)	M3: Expressions and Equations (35 days)	M3: Similarity (25 days)	2nd QUARTER
	M4: Expressions and Equations (45 days)	M4: Percent and Proportional Relationships (25 days)	M4: Linear Equations (40 days)	3rd QUARTER
3rd TRIMESTER	M5: Area, Surface Area, and Volume Problems (25 days)	M5: Statistics and Probability (25 days)	M5: Examples of Functions from Geometry (15 days)	
	M6: Statistics (25 days)	M6: Geometry (35 days)	M6: Linear Functions (20 days)	4th QUARTER
			M7: Introduction to Irrational Numbers Using Geometry (35 days)	

Key:	Number	Geometry	Ratios and Proportions	Expressions and Equations	Statistics and Probability	Functions

*The columns indicating trimesters and quarters are provided to give you a rough guideline. Please use your own pacing considerations based on the specific dates of your academic calender.

Figure 2.2 Grades 6–8 Year-Long Curriculum Map: A Story of Ratios

exponential functions, model projectiles and riverbeds using polynomial functions, model pistons using sinusoidal functions, and calculate the length of an on-off sequence used by computers to encode 256 characters using logarithmic functions.

Few U.S. textbooks paint mathematics as a dynamic, unfolding tale. They instead prioritize teaching procedures and employ a spiraling approach, in which topics are partially taught and then returned to—sometimes years later—with the unrealistic expectation that students will somehow connect the dots. But teaching procedures as skills without a rich context is ineffective. Students can too easily forget procedures and will fail if they do not have deeper, more concrete knowledge from which they can draw.

A *Story of Functions* weaves the mathematical concepts, applications, and fluency of the high school story together by relating everything students do back to the theme of studying functions. Algebra II students see how their current work with sinusoidal functions relates back to the work they did with polynomial functions; the processes used to study the two types of functions are the same even though the types of functions are different. This work, in turn, helps them focus on the differences between the properties of sinusoidal functions and the properties of polynomial functions.

Grade 9 – Algebra I	Grade 10 – Geometry	Grade 11 – Algebra II	Grade 12 – Precalculus
M1: Relationships Between Quantities and Reasoning with Equations and Their Graphs (40 days)	M1: Congruence, Proof, and Constructions (45 days)	M1: Polynomial, Rational, and Radical Relationships (45 days)	M1: Complex Numbers and Transformations (41 days)
M2: Descriptive Statistics (25 days)	M2: Similarity, Proof, and Trigonometry (45 days)	M2: Trigonometric Functions (20 days)	M2: Vectors and Matrices (36 days)
M3: Linear and Exponential Functions (35 days)	M3: Extending to Three Dimensions (15 days)	M3: Exponential and Logarithmic Functions (45 days)	M3: Rational and Exponential Functions (26 days)
M4: Polynomial and Quadratic Expressions, Equations, and Functions (30 days)	M4: Connecting Algebra and Geometry through Coordinates (20 days)	M4: Inferences and Conclusions from Data (40 days)	M4: Trigonometry (21 days)
M5: A Synthesis of Modeling with Equations and Functions (20 days)	M5: Circles with and Without Coordinates (25 days)		M5: Probability and Statistics (26 days)
30 days allotted for review and examinations**	30 days allotted for review and examinations**	30 days allotted for review and examinations**	30 days allotted for review and examinations**

1st TRIMESTER | 2nd TRIMESTER | 3rd TRIMESTER

1st QUARTER | 2nd QUARTER | 3rd QUARTER | 4th QUARTER

Key:	Number and Quantity and Modeling	Geometry and Modeling	Algebra and Modeling	Statistics and Probability and Modeling	Functions and Modeling

*The columns indicating trimesters and quarters are provided to give you a rough guideline. Please use your own pacing considerations based on the specific dates of your academic calender.
**Note that each high school grade accounts for 150 instructional days, allowing 30 days for teachers to address concerns and administer end of course exams.

Figure 2.3 Grades 9–12 Year-Long Curriculum Map: A Story of Functions

THE SIGNIFICANCE OF FUNCTIONS

Functions are the central objects in most fields of modern mathematics. They are also one of the fundamental tools that scientists use to model physical phenomena. A function is a correspondence between two sets in which each input element of the first set corresponds to one and only one output element of the second set. This definition generalizes the notion of proportional and linear relationships that students studied in middle school. In those grades, these relationships were represented by equations in which there was an independent variable (usually denoted by x) and a dependent variable (usually denoted by y). In high school, a linear relationship is represented as a function f from the set of real numbers to the set of real numbers by $f(x) = mx + b$ (where m is the rate of change and b is the initial value). It is then generalized to exponential, polynomial, rational, trigonometric, and logarithmic functions throughout high school.

As new concepts are introduced in high school, the overarching theme of functions remains the same. For example, students study the properties of logarithms (e.g., $\log ab = \log a + \log b$) by graphing functions such as $f(x) = \log 3 + \log x$ and $g(x) = \log 3x$, noticing the graphs are the same, and then proving conjectures about the properties of logarithms—just as they did when they were discovering relationships between sine and cosine functions earlier. Thus they build on their knowledge in new but analogous ways. The following paragraphs describe a few examples of major topics (in addition to those mentioned previously) in high school mathematics and how functions tie them together:

Sequences

A sequence is usually thought of as an ordered list of numbers, but it really is a function whose domain is contained within the set of nonnegative integers. Because the domain of a sequence is discrete, sequences are "simplified functions" that are very useful precursors to studying the "full version" functions. For example, the exponential function $f(x) = 2^x$ generates the sequence {1, 2, 4, 8, 16, . . .}, which carefully avoids having to discuss what the values of $2^{\frac{1}{2}}$ or 2^{π} are before students are ready. We use sequences in Algebra I to show how linear and exponential functions are like each other. (Linear functions have a constant rate of change, and exponential functions have a constant percent rate of change.) More important, we use sequences to show how linear and exponential functions differ.

Solving Equations

It is very common in modeling applications to compare values of two functions, f and g, and ask, "For what number(s) x is $f(x) = g(x)$ true?" For example, if $f(x) = x^2 - 3$ and $g(x) = 5x$, the question reduces to solving the equation $x^2 - 3 = 5x$. Students were introduced to solving linear and simple quadratic equations in middle school, but it is modeling with exponential, trigonometric, and logarithmic functions in high school that provides the rich context in which students generate new types of equations and learn the value of solving those equations.

Geometry

Geometry in high school is the study of geometric transformations—that is, *functions* that map points and figures in the plane to new points and figures in the plane. The most important geometric transformations are translations, reflections, rotations, and dilations. Translations, reflections, and rotations are functions that preserve distances, leading to the notion of congruence and theorems about congruent triangles. Dilations are functions that preserve the "shape" of figures—that is, dilations map figures to similar figures. It is through the study of geometric transformations that students generalize triangle congruence and similarity to any figure or graph in the plane (not just triangles).

Transformations of Functions

In Algebra I, students use geometric transformations to transform graphs of functions in exactly the same way they did for triangles. For example, by studying the graph of $y = f(x)$ and the graph of $y = f(x - 3)$, students see that the graph of $y = f(x - 3)$ is a translation of the graph of $y = f(x)$ by 3 units to the right in the Cartesian plane. They continue to use translations and reflections to study properties of the different types of functions throughout high school. In this way, geometry and algebra inform each other through the use of functions.

Calculus

Calculus is the study of differentiation and integration of *functions*. It is absolutely essential that students are comfortable with the main types of functions before starting a calculus

course. A *Story of Functions* was specifically designed to prepare high school students to take Calculus in their senior year or as a first-year college course. In particular, the functions and mathematical definitions used throughout A *Story of Functions* are the same as those used in the most popular calculus textbooks—every effort was made to prepare students for a smooth transition to Calculus and other advanced mathematics courses in college.

HOW A STORY OF FUNCTIONS ALIGNS WITH THE INSTRUCTIONAL SHIFTS

A *Story of Functions* is structured around the essential Instructional Shifts needed to implement the new college- and career-ready standards. These principles, articulated as three shifts (focus, coherence, and rigor), help educators understand what is required to implement the necessary changes. Rigor involves fluency, conceptual understanding, and application—and all three are achieved with a dual-intensity emphasis on practicing and understanding.

SHIFT 1: FOCUS

"**Focus deeply on the major work of each grade so that students can gain strong foundations.**"[1]

The Publishers' Criteria states: "Focus in high school is important in order to prepare students for college and careers."[2] Evidence of focus is seen in A *Story of Functions* through the integral use of the Partnership for Assessment of Readiness for College and Careers (PARCC) Content Emphases to focus on the major work of the course. Each module contains Focus Standards clearly stating the clusters of standards that are emphasized in the material. As noted in the Publishers' Criteria, "a college-ready curriculum . . . should devote the majority of students' time to building the particular knowledge and skills that are most important as prerequisites for a wide range of college majors, postsecondary programs, and careers."[3] Assessments concentrate on the work of the course and do not test students on content from prior courses or grade levels.

Foundational Standards are identified in each module so that teachers are empowered to assess and address gaps in prerequisite understanding as they begin the module. These standards are clearly distinguished from the major work of the course so that there is no question for teachers and students what their specific responsibilities are for the current year.

Student Outcomes are provided at the start of each lesson. These outcomes clarify expectations and are clearly shaped by the standards. Lessons allow students to work extensively with problems and exercises that are course appropriate.

SHIFT 2: COHERENCE

"**Learning is carefully connected across grades so that students can build new understanding onto foundations built in previous years.**"[4]

A *Story of Functions* is not a random collection of topics. Rather, the modules and topics in the curriculum are woven through the progressions of the standards. A *Story of Functions* carefully prioritizes and sequences the standards with a deliberate emphasis on mastery of major cluster standards. Modules connect supporting content to the major work of the

course in meaningful ways that enhance coherence. This meticulous sequencing enables students to transfer their mathematical knowledge and understanding to new, increasingly challenging concepts.

In each module, the Table of Contents shows how topics are aligned with standards to create an instructional sequence that is organized precisely to build on previous learning and to support future learning.

The Module Overview and Topic Overview narratives outline the instructional path, as do the Student Outcomes listed at the beginning of each lesson. The sequence of problems in the material is structured to help teachers analyze the mathematics for themselves and to help them differentiate instruction: As students advance from simple concepts to more complex, the different problems provide opportunities for teachers to either (1) break problems down for students struggling with a next step or (2) stretch problems out for those hungry for greater challenges. Modules include problems and exercises that connect two or more clusters in a domain or two or more domains in a course in meaningful ways, further developing coherence.

Foundational Standards assist teachers and students in relating course concepts explicitly to prior knowledge from previous courses or grades. New learning is built on an existing foundation of common knowledge shared by students, allowing connections to be made between new and previously learned content.

SHIFT 3: RIGOR

"**In major topics pursue: conceptual understanding, procedural skill and fluency, and application with equal intensity.**"[5]

The three-pronged nature of rigor undergirds a main theme of the Publishers' Criteria. Each of the three components of rigor—fluency, deep understanding, and application—must drive instruction with equal intensity for students to meet the standards' rigorous expectations.

Lessons provide problems designed to develop deep conceptual understanding, connect the content with mathematical and real-world problems, and cultivate fluency of newly developed skills. The distribution of tasks in each of the rigor categories is not prescribed but is driven by the content itself.

Conceptual Understanding

"**Students must be able to access concepts from a number of perspectives in order to see math as more than a set of mnemonics or discrete procedures.**"[6]

Conceptual understanding requires far more than performing discrete and often disjointed procedures to determine an answer. Students must not only learn mathematical content but also be able to access that knowledge from numerous vantage points and communicate about the process. In A *Story of Functions*, students use writing and speaking to solve mathematical problems, reflect on their learning, and analyze their thinking. The lessons and problem sets frequently require students to write solutions to word problems. Thus students learn to express their understanding of concepts and articulate their thought processes through writing. Similarly, students learn to verbalize the patterns and connections

between the current lesson and their previous learning, in addition to listening to and debating their peers' perspectives. The goal is to interweave the learning of new concepts with reflection time into students' everyday math experience.

At the module level, *sequence is everything*. Standards within the module and modules throughout the year carefully build to ensure that students have the requisite understanding to fully access new learning goals and integrate them into their developing schemas of understanding. The very deliberate progression of the material follows the critical instructional areas outlined in the introduction of the standards for each course.

Procedural Skill and Fluency

"**The Standards call for speed and accuracy in calculation. Students are given opportunities to practice core functions . . . so that they have access to more complex concepts and procedures.**"[7]

A *Story of Functions* provides students with ample opportunities to practice procedural skills, but teaching procedures without a rich context is ineffective. Students can too easily forget procedures and will fail if they do not have deeper, more concrete knowledge from which they can draw. A *Story of Functions* weaves the mathematical concepts, applications, and fluency of the high school story together by relating everything students do back to the theme of studying functions. Problem sets integrated into lessons for use in class and at home assist students on their path to fluency.

Application

"**The Standards call for students to use math flexibly for applications in problem-solving contexts.**"[8]

A *Story of Functions* is designed to help students understand how to choose and apply mathematics concepts to solve problems. To achieve this, the modules include diagrams that aid problem solving, interesting problems that encourage students to think quantitatively and creatively, and opportunities to model situations using mathematics. The goal is for students to come to see mathematics as connected to their environment, to other disciplines, and to the mathematics itself. Ranges of problems are presented within modules, topics, and lessons that serve multiple purposes:

- One-step word problems that help students understand the meaning of a particular concept.
- Multi-step word problems that support and develop instructional concepts and allow for the incorporation of multiple concepts into a single problem.
- Exploratory tasks designed to break potential habits of "rigid thinking."

Problems are designed so that there is a healthy mix of PARCC Type I, II, and III tasks. The three types of tasks outlined in PARCC are as follows:

Type I: Such tasks include computational problems, fluency exercises, conceptual problems, and applications, including one-step and multi-step word problems.

Type II: These tasks require students to demonstrate reasoning skills, justify their arguments, and critique the reasoning of their peers.

Type III: For these problems, students must model real-world situations using mathematics and demonstrate more advanced problem-solving skills.

HOW A STORY OF FUNCTIONS ALIGNS WITH THE STANDARDS FOR MATHEMATICAL PRACTICE

Like the Instructional Shifts, each Standard for Mathematical Practice is integrated into the design of *A Story of Functions.*

MP.1. Make sense of problems and persevere in solving them.

An explicit way in which the curriculum integrates this standard is through its commitment to consistently engage students in solving multi-step problems. For example, in Algebra I, students make sense of real-world problems involving linear and exponential growth. They analyze the critical components of the problem, presented as a verbal description, a data set, or a graph, and persevere in writing the appropriate function that describes the relationship between two quantities. In Module 3, students are sometimes required to write functions to represent data using a linear model and an exponential model and to compare how well each function equation represents the data. They select the most appropriate model and use it to make predictions. In Geometry, students work on a recurring problem throughout Module 4 in describing the motion of a robot bound within the room in which it sits. Through perseverance, students discover the slope criteria for perpendicular and parallel lines, the means to find the coordinates of a point dividing a line segment into two lengths in a given ratio, and the distance formula of a point from a line. In Algebra II, students solve rational and radical equations, which require them to consider the possibility of extraneous solutions. They also make sense of quadratic equations that do not contain real number solutions, coming to understand that the complex number system provides solutions to the equation $x^2 + 1 = 0$ and higher-degree equations. In Algebra II and Precalculus, students represent real-world situations with systems of equations, solve the systems using algebraic means or inverse matrix operations (in Precalculus), and examine the reasonableness of solutions.

Purposeful integration of a variety of problem types that range in complexity naturally invites students to analyze givens, constraints, relationships, and goals. Problems require students to organize their thinking, which necessitates critical self-reflection on the actions they take to problem-solve. On a more foundational level, concept sequence, activities, and lesson structure present information from a variety of novel perspectives. The question, “How can I look at this differently?” undergirds the organization of the curriculum, each of its components, and the design of problems.

MP.2. Reason abstractly and quantitatively.

An example of reasoning abstractly is seen in the analysis of graphs of functions in Module 1 of Algebra I. Students analyze graphs of non-constant rate measurements and infer from the shape of the graphs the quantities being displayed for a given situation and suggest appropriate units for the quantities. Throughout the lessons in Algebra I Module 5, students are required to decontextualize information provided as data or in a verbal description to analyze situations that can be represented using linear, quadratic, or

exponential models. Then they contextualize their work so that they can compare sets of data, make predictions, and evaluate claims. Similar opportunities are provided in Algebra II, where students decontextualize real-world situations to represent them with function equations. In Precalculus, students decontextualize data, representing it in matrix format, performing operations with matrices, and then interpreting the results in context.

In Geometry, students work with figures and their transformed images using symbolic representations and need to attend to the meaning of the symbolic notation to contextualize problems.

Quantitative reasoning is demonstrated explicitly in the statistics modules throughout *A Story of Functions*. For instance, in Algebra I Module 2, students use conditional relative frequencies to determine whether variables are associated, and they compare conditional probabilities in Algebra II Module 4 to determine whether two events are independent.

MP.3. Construct viable arguments and critique the reasoning of others.

Partner sharing is woven throughout lessons to create ongoing, frequent opportunities for students to develop this mathematical practice. Students use drawings, models and numeric representations, and precise language to make their learning and thinking understood by others. Discussions are also woven into the lessons, and these portions of the lessons include questions that require students to construct arguments, form conjectures, and use established properties to test them. The discussion times also allow students to present counterexamples to arguments generated by others.

An example of how the Algebra I *Eureka Math* curriculum supports this mathematical practice is found in Module 1. When students solve equations, they do so in an if-then format, where they are encouraged to construct arguments that justify how they progress from one step to another when they isolate a variable. This type of critical thinking is valuable for students as they progress to solving equations with complex solutions and where they will perform operations on equations that could introduce extraneous solutions.

In several modules in the *Eureka Math* Geometry curriculum, students are required to construct arguments about properties of objects. In Geometry Module 1, students construct arguments about unknown angle measures using formal proofs. In Module 2, they construct arguments to establish the criteria for similar figures in terms of dilations. Module 5 also requires students to construct arguments that justify constructions and to develop proofs related to circles.

In Algebra I Module 2, students examine the shape, center, and variability of a data distribution and use characteristics of the data distribution to communicate, in the form of a poster presentation, the answer to a statistical question. Students also have an opportunity to critique poster presentations made by other students. In Algebra II, students are required to construct viable arguments throughout Module 4 as they carry out hypothesis testing to compare a treatment group to a control group. This is continued in Precalculus, where students use laws of probability to justify decisions in a variety of real-world contexts, such as sports, financial investments, and medical care.

Throughout Module 1 in the Precalculus course, students critically analyze conjectures commonly formed by algebra students, and they are required to determine the validity of those arguments. Deciding on the validity of an argument focuses students on justification and argumentation as they work to decide when purported algebraic

identities do or do not hold. In cases where they decide that the given student work is incorrect, students work to develop the correct general algebraic results and justify them by reflecting on what they perceived as incorrect about the original student solution.

MP.4. Model with mathematics.

One interpretation of this standard is to ensure that students see that mathematics is useful. A *Story of Functions* contains modeling lessons throughout, where students apply a modeling process to a real-world situation. The specific modeling lessons are indicated in the Topic Overviews for each course.

Algebra I Module 5 develops students' modeling skills and requires students to be able to represent real-world situations with equations and graphs. A specific example involves modeling the cooling of an object with an exponential decay equation (this situation is revisited in Algebra II). In Module 4 of Geometry, students model the motion of a robot in the plane to determine the extent of motion within the bounds of a room and to move the robot to the location of the source of a beacon signal in the infinite plane. In Algebra II, students apply exponential models to explore mortgages and home loans, as well as investments that will help them pay off credit card debt and accrue savings over time. They also use trigonometric equations to model the motion of a Ferris wheel. In Precalculus, students use trigonometric functions to model the motion of waves. They use vectors to model the compressive forces distributed along stone arches to determine the effect of base column height and buttressing on the stability of the arch. They also apply their understanding of geometric representations of linear transformations to explore how three-dimensional objects are projected onto two-dimensional screens. In Module 4, they apply their understanding of trigonometry to determine the best viewing height of objects.

MP.5. Use appropriate tools strategically.

Using appropriate tools strategically must be interpreted broadly to include many options for students. In this curriculum, using appropriate tools strategically could include the use of tape diagrams, tabular models, transparencies, protractors, logarithm tables, random number generators, spreadsheet software, graphing applications, standard normal tables, rulers, and compasses.

Building students' independence with the use of models and tools is important, as is empowering students to determine when it is appropriate to use a specific tool. In Algebra I, students are introduced to the tabular model that can be applied to multiply polynomials and in the reverse to factor them. It is up to students to decide when this tool can be used efficiently and when it may be more efficient for them not to use this tool. Students need to decide for what sample size it makes sense to calculate measures of center and spread by hand and when it is more efficient to use a spreadsheet or graphing calculator. They also use graphing calculators to generate residual plots, which help them determine whether a function used to model a data set is appropriate. In Geometry, students need to decide what tools to use to complete constructions and geometric transformations—for instance, when they may need tracing paper to carry out a rigid transformation or when it might be most effective to explore the properties of objects using software. They need to determine when it is appropriate to calculate the exact measures of objects and when they may need to use a calculator to find a numerical estimate. In Algebra II, students may need to decide when it is beneficial to use graph paper to create the graph of a function versus using graphing software, which may enable them to more accurately produce graphs in three dimensions or graphs of

polynomials. In Precalculus, students may compare the advantages of carrying out simulations using manipulatives (e.g., coins or dice) versus creating a simulation using a graphing calculator or computer software. In Module 2, students use calculators and computer software to solve systems of three equations and three unknowns using matrices. Computer software is also used to help students visualize three-dimensional changes on a two-dimensional screen and in the creation of video games.

MP.6. Attend to precision.

Students are asked to attend to precision in all modules when they are precise in defining variables and when they use appropriate vocabulary and terminology when communicating with each other.

In Module 1 of Algebra I, students formalize descriptions of what they learned before (e.g., variables, solution sets, numerical expressions, algebraic expressions) as they build equivalent expressions and solve equations.

In Geometry, a clear application of this standard is found throughout the modules when students carry out constructions. Precision in carrying out the steps in a construction is critical in establishing and verifying geometric properties. This holds true for precisely measuring line segments and angles. In Module 3, students formalize definitions, using explicit language to define terms, such as *right rectangular prism*, that have been informal and more descriptive in earlier grades.

One example where precision is apparent in the Algebra II curriculum is in describing functions. The curriculum requires students to distinguish the attributes of a function, including its equation, inputs, outputs, domain, and range, and to identify the graph of the function and its features. It is important, for instance, for students to understand that transformations are applied to the graphs of functions and that $f(x) = 2x$ is an equation representing linear function f whose inputs are represented by x, whose outputs are represented by $2x$, and whose graph passes through the origin.

MP.7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. They can see algebraic expressions as single objects or as a composition of several objects. In Module 4 of Algebra I, two explicit cases illustrate how the *Eureka Math* curriculum supports students' making use of structure. First, in Lesson 2, several exercises require students to multiply binomials, and the structural similarities of the expanded products help students recognize patterns that enable them to factor quadratic trinomials into a product of two binomials. Several lessons in Topic C provide opportunities to explore the relationship between the structure of a function equation and the relationship to its graph. Specifically, students recognize how changing parameters in function equations affects the graphs of parent functions for quadratic functions, as well as cubic, square root, and cubed root functions.

The Algebra II curriculum provides expanded opportunities for students to recognize structure. Students apply their understanding of transformations of parent graphs from Algebra I to the graphs of exponential and logarithmic functions in Module 3 and graphs of trigonometric functions in Module 2. In Module 3, students extend the laws of exponents from integer exponents to rational and real number exponents. They connect how these laws are related to the properties of logarithms and understand how to rearrange an exponential equation into logarithmic form.

In Geometry, this practice is explicitly demonstrated in the way trigonometry is introduced and developed in Module 2. In Lesson 25, students complete an Exploratory Challenge that helps them recognize how the ratios of the length of the opposite side to the hypotenuse and the length of the adjacent side to the hypotenuse are the same for corresponding angles in similar triangles, which leads to the formalizing of trigonometric ratios. In addition, the theme of approximation in Module 3 is an interpretation of structure. Students approximate both area and volume for polyhedral regions. They must understand how and why it is possible to create upper and lower approximations of a figure's area or volume. The derivation of the volume formulas for cylinders, cones, and spheres and the use of Cavalieri's principle are also based entirely on understanding the structure and substructures of these figures.

In Precalculus, students recognize that the structure of matrices can help them identify the geometric transformations induced by them; for instance, they can identify matrices that induce pure dilations or pure rotations. They also recognize how to use the structure of Pascal's triangle to complete binomial expansions.

MP.8. Look for and express regularity in repeated reasoning.

One of the main themes of this mathematical practice is that mathematics is open to drawing conjectures from completing several related exercises, looking for regularity in both what you have done and in the results you obtain.

In Algebra I, students solve equations in one variable in several different formats. From the patterns they recognize, students look for general methods for solving a generic linear equation in one variable by replacing the numbers with letters: $ax + b = cx + d$. They can apply the method developed to rearrange formulas to isolate a variable of interest.

This practice is demonstrated in Module 4 of Geometry, specifically when students derive a formula used to calculate the midpoint of a line segment based on repeated reasoning from numeric examples.

In Algebra I and Algebra II, students recognize regularity in factoring patterns and develop identities to express that regularity, including identities for the difference of squares and cubes, the sum of squares, and the square of a sum and square of a difference. In Module 2 of Algebra II and Module 4 of Precalculus, students form conjectures about properties of trigonometric functions and develop formulas from the regularities they recognize, including angle sum and difference formulas for sine and cosine and the double angle formulas.

In summary, the Instructional Shifts and the Standards for Mathematical Practice help establish the mechanism for thoughtful sequencing and emphasis on key topics in *A Story of Functions*. It is evident that these pillars of the new standards combine to support *Eureka Math* with a structural foundation for the content. Consequently, *A Story of Functions* is artfully crafted to engage teachers and students alike while providing a powerful avenue for teaching and learning mathematics.

CHAPTER 3

Course Content Review

The Course Content Review begins with a list of modules developed to deliver instruction aligned to the standards for a given course. This introductory component is followed by three sections: the Summary of Year, the Rationale for Module Sequence, and the Alignment to the Standards and Placement of Standards in the Modules chart. The Summary of Year portion of each course includes three pieces of information:

- The purpose of the course
- The Recommended Fluencies for the course
- The Major Emphasis Clusters for the course

The Rationale for Module Sequence portion of each course provides a brief description of the instructional focus of each module for that course and explains the developmental sequence of the mathematics.

The Alignment chart for each course lists the standards that are addressed in each module of the course. Throughout the Alignment charts, when a cluster is included without a footnote, it is taught in its entirety; there are also times when footnotes are relevant to particular standards within a cluster. All standards for each course have been carefully included in the module sequence. Some standards are deliberately included in more than one module so that a strong foundation can be built over time.

The Course Content Review offers key information about course content and provides a recommended framework for grouping and sequencing topics and standards.

Sequence of Algebra II Modules Aligned with the Standards

Module 1: Polynomial, Rational, and Radical Relationships

Module 2: Trigonometric Functions

Module 3: Exponential and Logarithmic Functions

Module 4: Inferences and Conclusions from Data

Summary of Year

Building on their work with linear, quadratic, and exponential functions, students extend their repertoire of functions to include polynomial, rational, trigonometric, and logarithmic functions. Students work closely with the expressions that define the functions and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers and solving exponential equations using the properties of logarithms. The Standards for Mathematical Practice apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations.

Recommended Fluencies for Algebra II

- Divide polynomials with remainders by inspection in simple cases.
- See structure in expressions and use this structure to rewrite expressions (e.g., factoring, grouping).
- Translate between recursive definitions and closed forms for problems involving sequences and series.

Major Emphasis Clusters

The Real Number System

- Extend the properties of exponents to rational exponents

Seeing Structure in Expressions

- Interpret the structure of expressions
- Write expressions in equivalent forms to solve problems

Arithmetic with Polynomials and Rational Expressions

- Understand the relationship between zeros and factors of polynomials

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning
- Represent and solve equations and inequalities graphically

Interpreting Functions

- Interpret functions that arise in applications in terms of the context

Building Functions

- Build a function that models a relationship between two quantities

Making Inferences and Justifying Conclusions

- Make inferences and justify conclusions from sample surveys, experiments, and observational studies

RATIONALE FOR MODULE SEQUENCE IN ALGEBRA II

Mathematical Practices
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Module 1: In this module, students draw on analogies between polynomial arithmetic and base-10 computation, focusing on properties of operations, particularly the distributive property. Students connect the structure inherent in multi-digit whole number multiplication with multiplication of polynomials and similarly connect division of polynomials with long division of integers. Students identify zeros of polynomial functions, including complex zeros of quadratic functions. Through regularity in repeated reasoning, they make connections between zeros of polynomials and solutions of polynomial equations. A theme of this module is that just as the arithmetic of polynomial expressions is governed by the same rules as the arithmetic of integers, the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Module 2: Building on their previous work with functions and on their work with trigonometric ratios and circles in Geometry, students extend trigonometric functions to all (or most) real numbers. To reinforce their understanding of these functions, students begin building fluency with the values of sine, cosine, and tangent at $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, and so on. Students make sense of periodic behavior as they model real-world phenomena with trigonometric functions. Students expand on their work with polynomial identities to establish and prove trigonometric identities.

Module 3: Students extend their work with exponential functions to include modeling with exponential and logarithmic functions and solving exponential equations with logarithms. In addition, students synthesize and generalize what they have learned about a variety of function families. They continue to explore (with appropriate tools) the effects of transformations on graphs of diverse functions, including functions arising in an application. They notice, by looking for general methods in repeated calculations, that transformations of a function always have the same effect on the graph regardless of the type of the original function. These observations lead students to make conjectures and to construct general principles about how transforming a function changes its graph. Students identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as “the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions” (see p. 72 of CCSSM) is at the heart of this module. In particular, through repeated opportunities working through the modeling cycle, students acquire the insight that the same mathematical or statistical structure can sometimes model seemingly different situations.

Module 4: In this module, students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data, including sample surveys, experiments, and simulations; they also identify the role that randomness and careful design play in the conclusions that can be drawn. Students create theoretical and experimental probability models following the modeling cycle. They compute and interpret probabilities from those models for compound events, attending to mutually exclusive events, independent events, and conditional probability.

ALIGNMENT TO THE STANDARDS AND PLACEMENT OF STANDARDS IN THE MODULES

Module and Approximate Number of Instructional Days	Standards Addressed in Algebra II Modules
Module 1: Polynomial, Rational, and Radical Relationships (45 days)	**Reason quantitatively and use units to solve problems.** N-Q.A.2[1] Define appropriate quantities for the purpose of descriptive modeling. **Perform arithmetic operations with complex numbers.** N-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real. N-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers. **Use complex numbers in polynomial identities and equations.** N-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions. **Interpret the structure of expressions.** A-SSE.A.2[2] Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$. **Understand the relationship between zeros and factors of polynomials.** A-APR.B.2[3] Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$. A-APR.B.3[4] Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial. **Use polynomial identities to solve problems.** A-APR.C.4 Prove[5] polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.* **Rewrite rational expressions.** A-APR.D.6[6] Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system. **Understand solving equations as a process of reasoning and explain the reasoning.** A-REI.A.1[7] Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. A-REI.A.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. **Solve equations and inequalities in one variable.** A-REI.B.4[8] Solve quadratic equations in one variable. b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b. **Solve systems of equations.** A-REI.C.6[9] Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. A-REI.C.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ *and the circle* $x^2 + y^2 = 3$. **Analyze functions using different representations.** F-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases★[10] c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. **Translate between the geometric description and the equation for a conic section.** G-GPE.A.2 Derive the equation of a parabola given a focus and directrix.

Module and Approximate Number of Instructional Days	Standards Addressed in Algebra II Modules
Module 2: Trigonometric Functions (20 days)	**Analyze functions using different representations.** F-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. **Extend the domain of trigonometric functions using the unit circle.** F-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle. F-TF.A.2[11] Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. **Model periodic phenomena with trigonometric functions.** F-TF.B.5[12] Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★ **Prove and apply trigonometric identities.** F-TF.C.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle. **Summarize, represent, and interpret data on two categorical and quantitative variables.** S-ID.B.6[13] Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.★ a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.
Module 3: Exponential and Logarithmic Functions (45 days)	**Extend the properties of exponents to rational exponents.** N-RN.A.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{1/3}$ *to be the cube root of* 5 *because we want* $(5^{1/3})^3 = 5^{(1/3)3}$ *to hold, so* $(5^{1/3})^3$ *must equal* 5. N-RN.A.2[14] Rewrite expressions involving radicals and rational exponents using the properties of exponents. **Reason quantitatively and use units to solve problems.** N-Q.A.2[15] Define appropriate quantities for the purpose of descriptive modeling. **Write expressions in equivalent forms to solve problems.** A-SSE.B.3[16] Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is* 15%. A-SSE.B.4[17] Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★ **Create equations that describe numbers or relationships.** A-CED.A.1[18] Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★ **Represent and solve equations and inequalities graphically.** A-REI.D.11[19] Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. *Include cases where* $f(x)$ *and/or* $g(x)$ *are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*★ **Understand the concept of a function and use function notation.** F-IF.A.3[20] Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ *for* $n \geq 1$.

(*Continued*)

Module and Approximate Number of Instructional Days	Standards Addressed in Algebra II Modules
	Interpret functions that arise in applications in terms of the context. F-IF.B.4[21] For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity*.★ F-IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function* h(n) *gives the number of person-hours it takes to assemble* n *engines in a factory, then the positive integers would be an appropriate domain for the function.*★ F-IF.B.6[22] Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★ **Analyze functions using different representations.** F-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. F-IF.C.8[23] Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.* F-IF.C.9[24] Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* **Build a function that models a relationship between two quantities.** F-BF.A.1 Write a function that describes a relationship between two quantities.★ a. Determine an explicit expression, a recursive process, or steps for calculation from a context.[25] b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*[26] F-BF.A.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★ **Build new functions from existing functions.** F-BF.B.3[27] Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. F-BF.B.4 Find inverse functions. a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x + 1)/(x - 1)$ *for* $x \neq 1$. **Construct and compare linear, quadratic, and exponential models and solve problems.** F-LE.A.2[28] Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★ F-LE.A.4[29] For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★ **Interpret expressions for functions in terms of the situation they model.** F-LE.B.5[30] Interpret the parameters in a linear or exponential function in terms of a context.★

Module and Approximate Number of Instructional Days	Standards Addressed in Algebra II Modules
Module 4: Inferences and Conclusions from Data (40 days)	**Summarize, represent, and interpret data on a single count or measurement variable.** S-ID.A.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★ **Understand and evaluate random processes underlying statistical experiments.** S-IC.A.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.★ S-IC.A.2 Decide if a specified model is consistent with results from a given date-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability* 0.5. *Would a result of 5 tails in a row cause you to question the model?*★ **Make inferences and justify conclusions from sample surveys, experiments, and observational studies.** S-IC.B.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.★ S-IC.B.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★ S-IC.B.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.★ S-IC.B.6 Evaluate reports based on data.★ **Understand independence and conditional probability and use them to interpret data.** S-CP.A.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★ S-CP.A.2 Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★ S-CP.A.3 Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.★ S-CP.A.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*★ S-CP.A.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*★ **Use the rules of probability to compute probabilities of compound events in a uniform probability model.** S-CP.B.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.★ S-CP.B.7 Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model.★

EXTENSIONS TO THE ALGEBRA II COURSE

The (+) standards below are included in the Algebra II course to provide coherence to the curriculum. They can be used to effectively extend a topic or to introduce a theme/concept that will be fully covered in the Precalculus course.

Module 1. Students will be working with zeros of polynomials in this module, which offers teachers an opportunity to introduce Standard N-CN.C.9.	**Use complex numbers in polynomial identities and equations.** N-CN.C.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$. N-CN.C.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.
A major theme of the module is A-APR.D.7. Teachers should continually remind students of the connections between rational expressions and rational numbers as students add, subtract, multiply, and divide rational expressions.	**Rewrite rational expressions.** A-APR.D.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.
Module 2. In F-TF.A.3, students begin fluency exercises with trigonometric ratios of the special angles.	**Extend the domain of trigonometric functions using the unit circle.** F-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.
Teachers present proofs of formulas in F-TF.C.9. Students use the formulas in Algebra II; they prove the formulas in Precalculus.	**Prove and apply trigonometric identities.** F-TF.C.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

CHAPTER 4

Curriculum Design

Curriculum design details the approach to modules, lessons, and assessment in *A Story of Functions*. This chapter describes the key elements in the modules, as well as the role each plays in successful implementation of the new standards. We provide a wealth of information about how to achieve the components of instructional rigor (fluency, concept development, and application) that the standards demand.

APPROACH TO MODULE STRUCTURE

One of the first steps in designing a curriculum to implement the standards is to determine the grouping and sequencing of the standards at each grade level or course. This process should be well thought out, and decisions should be based on sound rationale. It is imperative that standards for a particular grade level or course be organized appropriately and addressed adequately.

Making key information in each module or unit of study available to classroom teachers helps to ensure successful implementation of the standards. The authors of the curriculum included numerous features to assist teachers not only in their day-to-day classroom activities but also in developing their understanding of the mathematics content and the standards.

Each module has four primary parts: Module Overview, Topic Overviews, Lessons, and Assessments:

- The Module Overview, rich with valuable information, introduces the key components of each module. It outlines the progression of the lessons from the beginning of the module to the end. The components of the overview are as follows:
    - The Table of Contents serves as a pacing guide and indicates the topics in the module along with the associated standards. The lessons in each topic are listed with each estimated to take one 45-minute instructional period. Also noted are the timing of the Mid-Module and End-of-Module Assessments.
    - The opening narrative explains the progression of the mathematics through the module topic by topic.

- Focus Standards are the major standards that the module targets.
- Foundational Standards comprise prerequisite knowledge and support the Focus Standards. These include standards addressed prior to the module that are essential for student learning and understanding. This section can be helpful in preparing for teaching lessons and for addressing any gaps that might crop up, especially during the first couple of years of implementation.
- Focus Standards for Mathematical Practice act as a guide to create a well-rounded, standards-aligned classroom environment. This module element highlights which of the eight Standards for Mathematical Practice are a focus of the module and explains how each is addressed in the module.
- Terminology consists of both new and recently introduced terms and familiar terms and symbols. Definitions of the terms are included in this section.
- Suggested Tools and Representations provides teachers with a list of the models, manipulatives, diagrams, and so forth that are recommended to teach the content of the module.
- The Assessment Summary gives key information about the assessments of the module, including where in the module they are given and what standards are addressed.

- Each Topic Overview lists the Focus Standards as well as the lessons associated with the topic. Each also provides a narrative similar to that found in the Module Overview but with information regarding specific lessons within the topic.
- Lessons and assessments are the remaining primary parts of the module. The sections that follow detail these module components.

APPROACH TO LESSON STRUCTURE

Fluency, concept development, and application, all components of instructional rigor, are demanded by the new standards and should be layered into daily lessons to help guide students through the mathematics. Lessons must be structured to incorporate the development of conceptual understanding, procedural skills, and problem solving. The time spent on each component of rigor in a daily lesson should vary between lessons and is guided by the rigor emphasized in the standard(s) the lesson is addressing. Ideally, these components are taught through the deliberate progression of material. During the early grades, this progression typically moves from concrete to pictorial to abstract. As students progress through the grades and into middle and high school, the emphasis remains on moving from simple to complex, although the dependency on concrete and pictorial representations diminishes.

All stages of instruction ought to be designed to help students reach higher and higher levels of understanding. The lesson design in A *Story of Functions* reflects this ideal and is an exemplary model of how to achieve the demands that the standards necessitate. Through a balanced approach to lesson design, A *Story of Functions* supports the development of an increasingly complex understanding of the mathematical concepts and topics within the standards. This type of balanced approach to lesson design naturally reveals patterns and connections among concepts, tools, strategies, and real-world applications.

Daily lessons in the modules are designed to be 45 minutes in duration and are designated as one of the four lesson types described in the next paragraphs. The lesson type designation is given using one of the four icons shown (Figure 4.1) and provides the teacher with a general idea of what to expect from the structure of the lesson.

1. **Problem Set**
   The teacher and students work through examples and complete exercises to develop or reinforce a concept.

2. **Socratic**
   The teacher leads students in a conversation to develop a specific concept or proof.

3. **Exploration**
   Students work independently or in small groups on a challenging problem followed by a debrief to clarify, expand, or develop math knowledge.

4. **Modeling Cycle**
   Students practice all or part of the modeling cycle with real-world or mathematical problems that are either well- or ill-defined.

Figure 4.1

In the Problem Set Lesson, teachers and students work through a sequence of examples and exercises to develop or reinforce a concept. The majority of the class period is spent alternating between the teacher working through examples followed by students completing exercises either individually or in pairs.

In the Socratic Lesson, the teacher leads students in a conversation with the aim of developing a specific concept or proof. This approach is often used when conveying an idea that students cannot learn or discover on their own. The teacher asks guiding questions to pull information from the students and draw them into the discussion. Other activities may occur during the lesson in the minutes that remain. For example, a fluency activity may open the lesson, or there may be a debrief or application problem at the end of the lesson.

The Exploration Lesson involves students working independently or in small groups for a sustained period of time on one or more exploratory challenges, followed by a debrief with the goal of clarifying, expanding on, or developing a concept, definition, theorem, or proof. Because of the rigorous nature of the exercise, allowing students to collaborate in pairs or groups should be a consideration. The class comes back together to discuss the challenge, draw conclusions, and consolidate understandings.

In the Modeling Cycle Lesson, students and teacher work through the modeling cycle to complete an application problem over one or two days. The lesson may involve students practicing all or part of the modeling cycle. In Grades 6–8, a reduced form of the modeling cycle is typical, whereas in Grades 9–12, the modeling cycle in its entirety is generally incorporated into lessons. The typical problem that students encounter in this lesson type is ill-defined and has a real-world context. Students are likely to work in groups on these types of problems, but teachers may want students to work for a period of time individually before collaborating with their group members.

The lessons incorporate guidance for teachers in several ways. Narratives appear throughout the lessons to help teachers understand the content being delivered and how to

effectively execute the lesson. Scaffolding boxes suggest ways to address the needs of diverse learners at specific points in the lessons, and other notes to the teacher give insight or required background material for the lesson.

The Instructional Shifts require that the three components of rigor be taught with equal intensity. Although this does not mean an “equal amount of time for each component per day,” any alternative lesson structure needs to meet the high expectations demanded by the standards for all three components of rigor in the major work of each grade or course.

SAMPLE LESSON

The following lesson provides an example of one of the four lesson types described previously, the Problem Set. This is Lesson 17 from Algebra II Module 3, titled Graphing the Logarithm Function.

Lesson 17: Graphing the Logarithm Function

Student Outcomes

- Students graph the functions $f(x) = \log(x)$, $g(x) = \log_2(x)$, and $h(x) = \ln(x)$ by hand and identify key features of the graphs of logarithmic functions.

Lesson Notes

In this lesson, students work in pairs or small groups to generate graphs of $f(x) = \log(x)$, $g(x) = \log_2(x)$, or $h(x) = \log_5(x)$. Students compare the graphs of these three functions to derive the key features of graphs of general logarithmic functions for bases $b > 1$. Tables of function values are provided so that calculators are not needed in this lesson; all graphs should be drawn by hand. Students relate the domain of the logarithmic functions to the graph in accordance with **F-IF.B.5**. After the graphs are generated and conclusions drawn about their properties, students use properties of logarithms to find additional points on the graphs. Continue to rely on the definition of the logarithm, which was stated in Lesson 8, and properties of logarithms developed in Lessons 12 and 13:

LOGARITHM: If three numbers, L, b, and x are related by $x = b^L$, then L is the *logarithm base* b of x, and we write $\log_b(x)$. That is, the value of the expression $L = \log_b(x)$ is the power of b needed to obtain x. Valid values of b as a base for a logarithm are $0 < b < 1$ and $b > 1$.

Classwork

Opening (1 minute)

Divide the students into pairs or small groups; ideally, the number of groups formed is a multiple of three. Assign the function $f(x) = \log(x)$ to one-third of the groups, and refer to these groups as the 10-team. Assign the function $g(x) = \log_2(x)$ to the second third of the groups, and refer to these groups as the 2-team. Assign the function $h(x) = \log_5(x)$ to the remaining third of the groups, and refer to these groups as the 5-team.

Scaffolding:

- Struggling students may benefit from watching the teacher model the process of plotting points.
- Consider assigning struggling students to the 2-team because the function values are integers.
- Alternatively, assign advanced students to the 2-team and ask them to generate the graph of $y = \log_2(x)$ without the given table.

Opening Exercise (8 minutes)

While student groups are creating graphs and responding to the prompts that follow, circulate and observe student work. Select three groups to present their graphs and results at the end of the exercise.

Figure 4.2

A STORY OF FUNCTIONS — Lesson 17 M3
ALGEBRA II

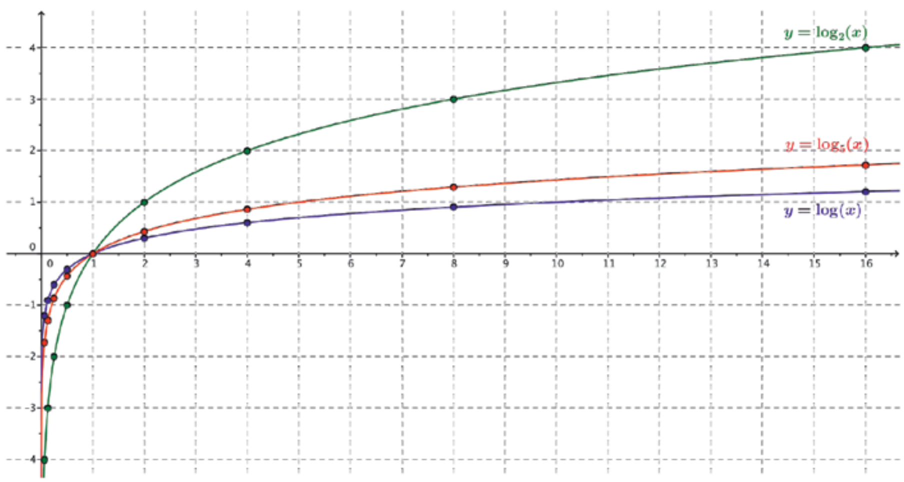
Opening Exercise

Graph the points in the table for your assigned function $f(x) = \log(x)$, $g(x) = \log_2(x)$, or $h(x) = \log_5(x)$ for $0 < x \leq 16$. Then, sketch a smooth curve through those points and answer the questions that follow.

10-team $f(x) = \log(x)$	
x	$f(x)$
0.0625	-1.20
0.125	-0.90
0.25	-0.60
0.5	-0.30
1	0
2	0.30
4	0.60
8	0.90
16	1.20

2-team $g(x) = \log_2(x)$	
x	$g(x)$
0.0625	-4
0.125	-3
0.25	-2
0.5	-1
1	0
2	1
4	2
8	3
16	4

5-team $h(x) = \log_5(x)$	
x	$h(x)$
0.0625	-1.72
0.125	-1.29
0.25	-0.86
0.5	-0.43
1	0
2	0.43
4	0.86
8	1.29
16	1.72



a. **What does the graph indicate about the domain of your function?**

The domain of each of these functions is the positive real numbers, which can be stated as $(0, \infty)$.

b. **Describe the x-intercepts of the graph.**

There is one x-intercept at 1.

c. **Describe the y-intercepts of the graph.**

There are no y-intercepts of this graph.

d. **Find the coordinates of the point on the graph with y-value 1.**

For the 10-team, this is $(10, 1)$. For the 2-team, this is $(2, 1)$. For the 5-team, this is $(5, 1)$.

Figure 4.2 (Continued)

e. **Describe the behavior of the function as $x \to 0$.**

As $x \to 0$, the function values approach negative infinity; that is, $f(x) \to -\infty$. The same is true for the functions g and h.

f. **Describe the end behavior of the function as $x \to \infty$.**

As $x \to \infty$, the function values slowly increase. That is, $f(x) \to \infty$. The same is true for the functions g and h.

g. **Describe the range of your function.**

The range of each of these functions is all real numbers, $(-\infty, \infty)$.

h. **Does this function have any relative maxima or minima? Explain how you know.**

Since the function values continue to increase, and there are no peaks or valleys in the graph, the function has no relative maxima or minima.

Presentations (5 minutes)

Select three groups of students to present each of the three graphs, projecting each graph through a document camera or copying the graph onto a transparency sheet and displaying on an overhead projector. Ask students to point out the key features they identified in the Opening Exercise on the displayed graph. If students do not mention it, emphasize that the long-term behavior of these functions is they are always increasing, although very slowly.

As representatives from each group make their presentations, record their findings on a chart. This chart can be used to help summarize the lesson and to later display in the classroom.

	$f(x) = \log(x)$	$g(x) = \log_2(x)$	$h(x) = \log_5(x)$
Domain of the Function	$(0, \infty)$	$(0, \infty)$	$(0, \infty)$
Range of the Function	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$
x-intercept	1	1	1
y-intercept	*None*	*None*	*None*
Point with y-value 1	$(10, 1)$	$(2, 1)$	$(5, 1)$
Behavior as $x \to 0$	$f(x) \to -\infty$	$g(x) \to -\infty$	$h(x) \to -\infty$
End Behavior as $x \to \infty$	$f(x) \to \infty$	$g(x) \to \infty$	$h(x) \to \infty$

Discussion (5 minutes)

Debrief the Opening Exercise by asking students to generalize the key features of the graphs $y = \log_b(x)$. If possible, display the graph of all three functions $f(x) = \log(x)$, $g(x) = \log_2(x)$, and $h(x) = \log_5(x)$ together on the same axes during this discussion.

We saw in Lesson 5 that the expression 2^x is defined for all real numbers x; therefore, the range of the function $g(x) = \log_2(x)$ is all real numbers. Likewise, the expressions 10^x and 5^x are defined for all real numbers x, so the range of the functions f and h is all real numbers. Notice that since the range is all real numbers in each case, there must be logarithms that are irrational. We saw examples of such logarithms in Lesson 16.

Figure 4.2 (Continued)

- What are the domain and range of the logarithm functions?
  - *The domain is the positive real numbers, and the range is all real numbers.*
- What do the three graphs of $f(x) = \log(x)$, $g(x) = \log_2(x)$, and $h(x) = \log_5(x)$ have in common?
  - *The graphs all cross the x-axis at $(1, 0)$.*
  - *None of the graphs intersect the y-axis.*
  - *They have the same end behavior as $x \to \infty$, and they have the same behavior as $x \to 0$.*
  - *The functions all increase quickly for $0 < x < 1$, then increase more and more slowly.*
- What do you expect the graph of $y = \log_3(x)$ to look like?
  - *It will look just like the other graphs, except that it will lie between the graphs of $y = \log_2(x)$ and $y = \log_5(x)$ because $2 < 3 < 5$.*
- What do you expect the graph of $y = \log_b(x)$ to look like for any number $b > 1$?
  - *It will have the same key features of the other graphs of logarithmic functions. As the value of b increases, the graph will flatten as $x \to \infty$.*

Exercise 1 (8 minutes)

Keep students in the same groups for this exercise. Students plot points and sketch the graph of $y = \log_{\frac{1}{b}}(x)$ for $b = 10$, $b = 2$, or $b = 5$, depending on whether they are on the 10-team, the 2-team, or the 5-team. Then, students observe the relationship between their two graphs, justify the relationship using properties of logarithms, and generalize the observed relationship to graphs of $y = \log_b(x)$ and $y = \log_{\frac{1}{b}}(x)$ for $b > 0$, $b \neq 1$.

Exercises

1. Graph the points in the table for your assigned function $r(x) = \log_{\frac{1}{10}}(x)$, $s(x) = \log_{\frac{1}{2}}(x)$, or $t(x) = \log_{\frac{1}{5}}(x)$ for $0 < x \leq 16$. Then, sketch a smooth curve through those points, and answer the questions that follow.

10-team $r(x) = \log_{\frac{1}{10}}(x)$	
x	$r(x)$
0.0625	1.20
0.125	0.90
0.25	0.60
0.5	0.30
1	0
2	-0.30
4	-0.60
8	-0.90
16	-1.20

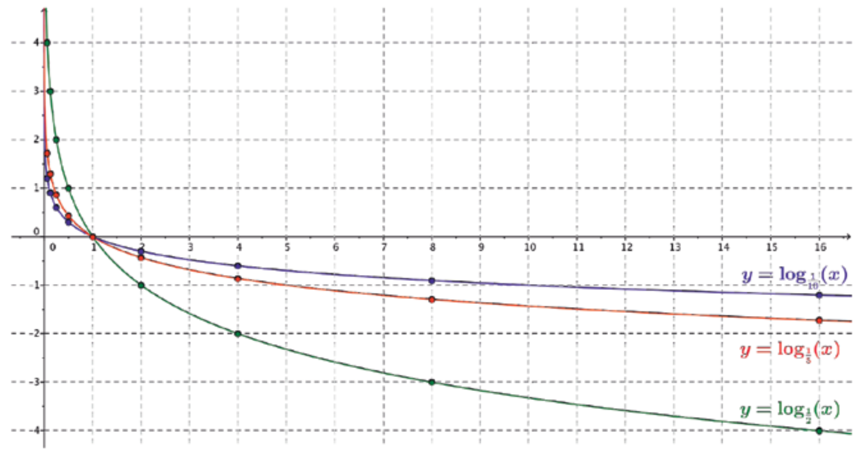
2-team $s(x) = \log_{\frac{1}{2}}(x)$	
x	$s(x)$
0.0625	4
0.125	3
0.25	2
0.5	1
1	0
2	-1
4	-2
8	-3
16	-4

e-team $t(x) = \log_{\frac{1}{5}}(x)$	
x	$t(x)$
0.0625	1.72
0.125	1.29
0.25	0.86
0.5	0.43
1	0
2	-0.43
4	-0.86
8	-1.29
16	-1.72

Figure 4.2 (Continued)

A STORY OF FUNCTIONS **Lesson 17** **M3**

ALGEBRA II



a. **What is the relationship between your graph in the Opening Exercise and your graph from this exercise?**

The second graph is the reflection of the graph in the Opening Exercise across the x-axis.

b. **Why does this happen? Use the change of base formula to justify what you have observed in part (a).**

Using the change of base formula, we have $\log_{\frac{1}{2}}(x) = \frac{\log_2(x)}{\log_2\left(\frac{1}{2}\right)}$.

Since $\log_2\left(\frac{1}{2}\right) = -1$, *we have* $\log_{\frac{1}{2}}(x) = \frac{\log_2(x)}{-1}$, *so* $\log_{\frac{1}{2}}(x) = -\log_2(x)$.

Thus, the graph of $y = \log_{\frac{1}{2}}(x)$ *is the reflection of the graph of* $y = \log_2(x)$ *across the x-axis.*

> *Scaffolding:*
> Students struggling with the comparison of graphs may find it easier to draw the graphs on transparent plastic sheets and compare them that way.

Discussion (4 minutes)

Ask students from each team to share their graphs results from part (a) of Exercise 1 with the class. During their presentations, complete the chart below.

	$r(x) = \log_{\frac{1}{10}}(x)$	$s(x) = \log_{\frac{1}{2}}(x)$	$t(x) = \log_{\frac{1}{5}}(x)$
Domain of the Function	$(0, \infty)$	$(0, \infty)$	$(0, \infty)$
Range of the Function	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$
x-intercept	1	1	1
y-intercept	None	None	None
Point with y-value -1	$(10, -1)$	$(2, -1)$	$(5, -1)$
Behavior as $x \to 0$	$r(x) \to \infty$	$s(x) \to \infty$	$t(x) \to \infty$
End Behavior as $x \to \infty$	$r(x) \to \infty$	$s(x) \to -\infty$	$t(x) \to -\infty$

Figure 4.2 (Continued)

Then proceed to hold the following discussion.

- From what we have seen of these three sets of graphs of functions, can we state the relationship between the graphs of $y = \log_b(x)$ and $y = \log_{\frac{1}{b}}(x)$, for $b \neq 1$?
  - *If $b \neq 1$, then the graphs of $y = \log_b(x)$ and $y = \log_{\frac{1}{b}}(x)$ are reflections of each other across the x-axis.*
- Describe the key features of the graph of $y = \log_b(x)$ for $0 < b < 1$.
  - *The graph crosses the x-axis at $(1, 0)$.*
  - *The graph does not intersect the y-axis.*
  - *The graph passes through the point $(b, -1)$.*
  - *As $x \to 0$, the function values increase quickly; that is, $f(x) \to \infty$.*
  - *As $x \to \infty$, the function values continue to decrease; that is, $f(x) \to -\infty$.*
  - *There are no relative maxima or relative minima.*

Exercises 2–3 (6 minutes)

Keep students in the same groups for this set of exercises. Students plot points and sketch the graph of $y = \log_b(bx)$ for $b = 10$, $b = 2$, or $b = 5$, depending on whether they are on the 10-team, the 2-team, or the 5-team. Then, students observe the relationship between their two graphs, justify the relationship using properties of logarithms, and generalize the observed relationship to graphs of $y = \log_b(x)$ and $y = \log_b(x)$ for $b > 0$, $b \neq 1$. If there is time at the end of these exercises, consider using GeoGebra or other dynamic geometry software to demonstrate the property illustrated in Exercise 3 below by graphing $y = \log_2(x)$, $y = \log_2(2x)$, and $y = 1 + \log_2(x)$ on the same axes.

Consider having students graph these functions on the same axes as used in the Opening Exercise.

2. In general, what is the relationship between the graph of a function $y = f(x)$ and the graph of $y = f(kx)$ for a constant k?

The graph of $y = f(kx)$ is a horizontal scaling of the graph of $y = f(x)$.

3. Graph the points in the table for your assigned function $u(x) = \log(10x)$, $v(x) = \log_2(2x)$, or $w(x) = \log_5(5x)$ for $0 < x \leq 16$. Then sketch a smooth curve through those points, and answer the questions that follow.

10-team $u(x) = \log(10x)$	
x	$u(x)$
0.0625	-0.20
0.125	0.10
0.25	0.40
0.5	0.70
1	1
2	1.30
4	1.60
8	1.90
16	2.20

2-team $v(x) = \log_2(2x)$	
x	$v(x)$
0.0625	-3
0.125	-2
0.25	-1
0.5	0
1	1
2	2
4	3
8	4
16	5

5-team $w(x) = \log_5(5x)$	
x	$w(x)$
0.0625	-0.72
0.125	-0.29
0.25	0.14
0.5	0.57
1	1
2	1.43
4	1.86
8	2.29
16	2.72

Figure 4.2 (Continued)

A STORY OF FUNCTIONS Lesson 17 M3

ALGEBRA II

MP.7

a. **Describe a transformation that takes the graph of your team's function in this exercise to the graph of your team's function in the Opening Exercise.**

The graph produced in this exercise is a vertical translation of the graph from the Opening Exercise by one unit upward.

b. **Do your answers to Exercise 2 and part (a) agree? If not, use properties of logarithms to justify your observations in part (a).**

The answers to Exercise 2 and part (a) do not appear to agree. However, because $\log_b(bx) = \log_b(b) + \log_b(x) = 1 + \log_b(x)$, the graph of $y = \log_b(bx)$ and the graph of $y = 1 + \log_b(x)$ coincide.

Closing (3 minutes)

Ask students to respond to these questions in writing or orally to a partner.

- In which quadrants is the graph of the function $f(x) = \log_b(x)$ located?
  - *The first and fourth quadrants*
- When $b > 1$, for what values of x are the values of the function $f(x) = \log_b(x)$ negative?
  - *When $b > 1$, $f(x) = \log_b(x)$ is negative for $0 < x < 1$.*
- When $0 < b < 1$, for what values of x are the values of the function $f(x) = \log_b(x)$ negative?
  - *When $0 < b < 1$, $f(x) = \log_b(x)$ is negative for $x > 1$.*
- What are the key features of the graph of a logarithmic function $f(x) = \log_b(x)$ when $b > 1$?
  - *The domain of the function is all positive real numbers, and the range is all real numbers. The x-intercept is 1, the graph passes through $(b, 1)$ and there is no y-intercept. As $x \to 0$, $f(x) \to -\infty$ quickly, and as $x \to \infty$, $f(x) \to \infty$ slowly.*
- What are the key features of the graph of a logarithmic function $f(x) = \log_b(x)$ when $0 < b < 1$?
  - *The domain of the function is the positive real numbers, and the range is all real numbers. The x-intercept is 1, the graph passes through $(b, -1)$, and there is no y-intercept. As $x \to 0$, $f(x) \to \infty$ quickly, and as $x \to \infty$, $f(x) \to -\infty$ slowly.*

Lesson Summary

The function $f(x) = \log_b(x)$ is defined for irrational and rational numbers. Its domain is all positive real numbers. Its range is all real numbers.

The function $f(x) = \log_b(x)$ goes to negative infinity as x goes to zero. It goes to positive infinity as x goes to positive infinity.

The larger the base b, the more slowly the function $f(x) = \log_b(x)$ increases.

By the change of base formula, $\log_{\frac{1}{b}}(x) = -\log_b(x)$.

Exit Ticket (5 minutes)

Figure 4.2 (Continued)

Name ____ Date ________________

Lesson 17: Graphing the Logarithm Function

Exit Ticket

Graph the function $f(x) = \log_3(x)$ without using a calculator, and identify its key features.

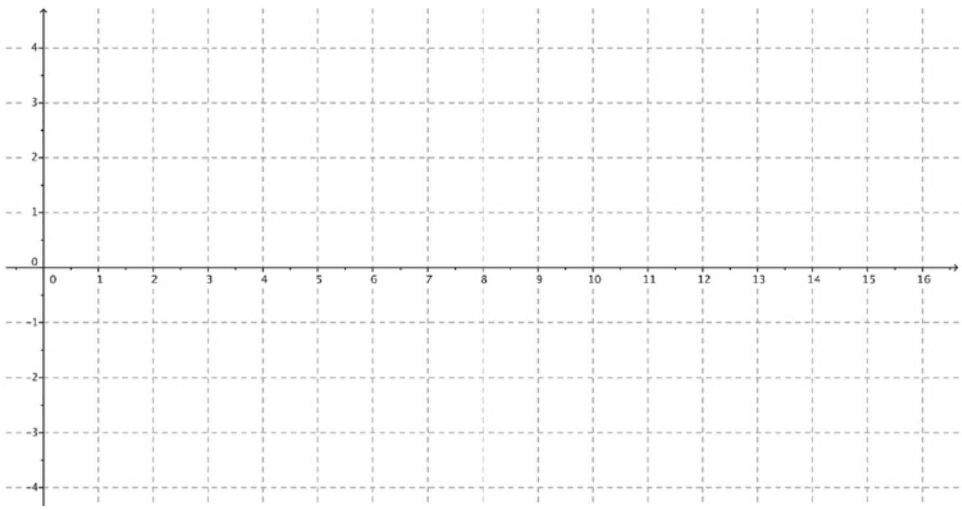
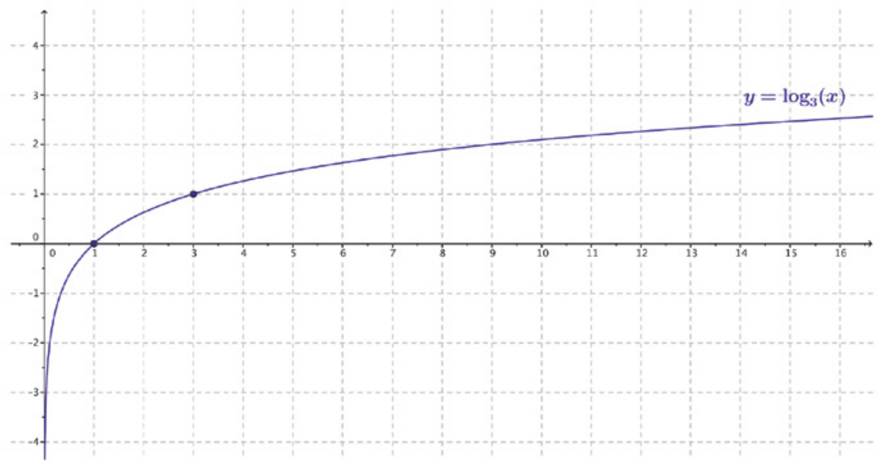


Figure 4.2 (Continued)

A STORY OF FUNCTIONS — Lesson 17 — M3
ALGEBRA II

Exit Ticket Sample Solutions

Graph the function $f(x) = \log_3(x)$ without using a calculator, and identify its key features.



Key features:

The domain is $(0, \infty)$.

The range is all real numbers.

End behavior:

As $x \to 0$, $f(x) \to -\infty$.

As $x \to \infty$, $f(x) \to \infty$.

Intercepts:

x-intercept: There is one x-intercept at 1.

y-intercept: The graph does not cross the y-axis.

The graph passes through $(3, 1)$.

Figure 4.2 (Continued)

Problem Set Sample Solutions

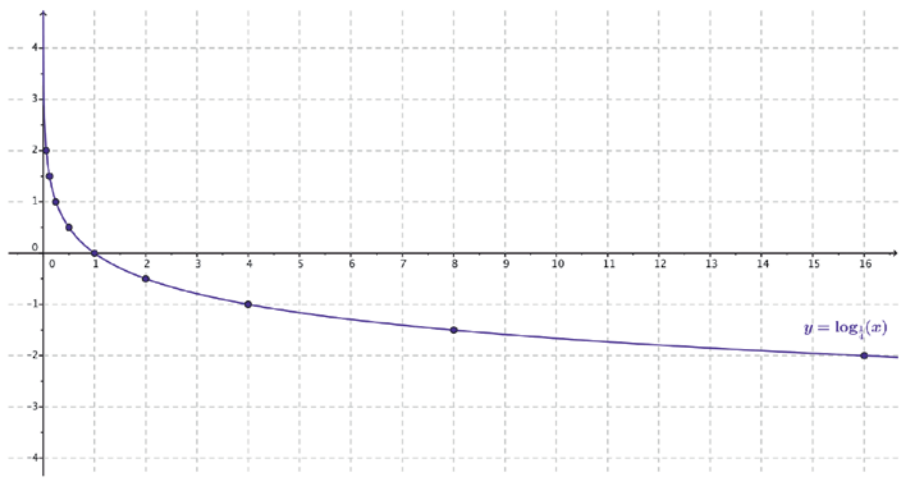
For the Problem Set, students need graph paper. They should not use calculators or other graphing technology, except where specified in extension Problems 11 and 12. In Problems 2 and 3, students compare different representations of logarithmic functions. Problems 4–6 continue the reasoning from the lesson in which students observed logarithmic properties through the transformations of logarithmic graphs.

Fluency problems 9–10 are a continuation of work done in Algebra I and have been placed in this lesson so that students recall concepts required in Lesson 19. Similar review problems occur in the next lesson.

1. The function $Q(x) = \log_b(x)$ has function values in the table at right.

a. Use the values in the table to sketch the graph of $y = Q(x)$.

x	$Q(x)$
0.1	1.66
0.3	0.87
0.5	0.50
1.00	0.00
2.00	-0.50
4.00	-1.00
6.00	-1.29
10.00	-1.66
12.00	-1.79



b. What is the value of b in $Q(x) = \log_b(x)$? Explain how you know.

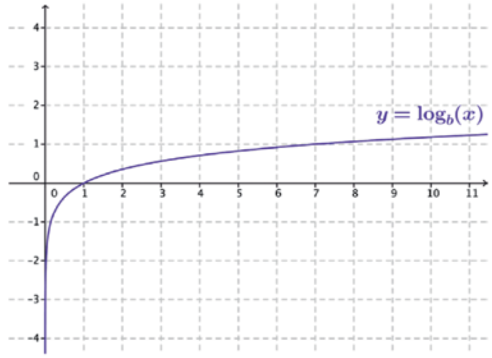
Because the point $(4, -1)$ is on the graph of $y = Q(x)$, we know $\log_b(4) = -1$, so $b^{-1} = 4$. It follows that $b = \frac{1}{4}$.

c. Identify the key features in the graph of $y = Q(x)$.

Because $0 < b < 1$, the function values approach ∞ as $x \to 0$, and the function values approach $-\infty$ as $x \to \infty$. There is no y-intercept, and the x-intercept is 1. The domain of the function is $(0, \infty)$, the range is $(-\infty, \infty)$, and the graph passes through $(b, 1)$.

Figure 4.2 (Continued)

2. Consider the logarithmic functions $f(x) = \log_b(x)$, $g(x) = \log_5(x)$, where b is a positive real number, and $b \neq 1$. The graph of f is given at right.



a. Is $b > 5$, or is $b < 5$? Explain how you know.

Since $f(7) = 1$, and $g(7) \approx 1.21$, the graph of f lies below the graph of g for $x \geq 1$. This means that b is larger than 5, so we have $b > 5$. (Note: The actual value of b is 7.)

b. Compare the domain and range of functions f and g.

Functions f and g have the same domain, $(0, \infty)$, and the same range, $(-\infty, \infty)$.

c. Compare the x-intercepts and y-intercepts of f and g.

Both f and g have an x-intercept at 1 and no y-intercepts.

d. Compare the end behavior of f and g.

As $x \to \infty$, both $f(x) \to \infty$ and $g(x) \to \infty$.

3. Consider the logarithmic functions $f(x) = \log_b(x)$ and $g(x) = \log_{\frac{1}{2}}(x)$, where b is a positive real number and $b \neq 1$. A table of approximate values of f is given below.

x	$f(x)$
$\frac{1}{4}$	0.86
$\frac{1}{2}$	0.43
1	0
2	-0.43
4	-0.86

a. Is $b > \frac{1}{2}$, or is $b < \frac{1}{2}$? Explain how you know.

Since $g(2) = -1$, and $f(2) \approx -0.43$, the graph of f lies above the graph of g for $x \geq 1$. This means that b is closer to 0 than $\frac{1}{2}$ is, so we have $b < \frac{1}{2}$. (Note: The actual value of b is $\frac{1}{5}$.)

b. Compare the domain and range of functions f and g.

Functions f and g have the same domain, $(0, \infty)$, and the same range, $(-\infty, \infty)$.

Figure 4.2 (Continued)

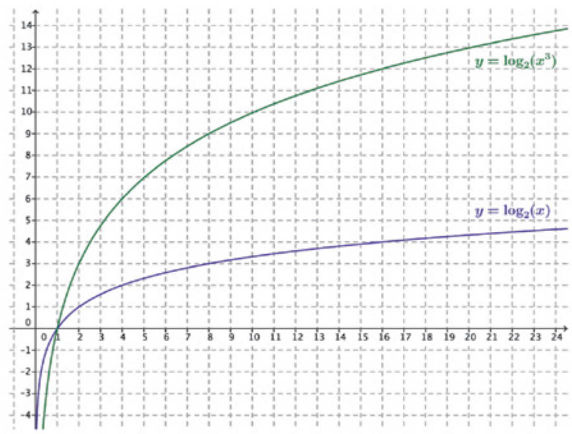
c. Compare the x-intercepts and y-intercepts of f and g.

Both f and g have an x-intercept at 1 and no y-intercepts.

d. Compare the end behavior of f and g.

As $x \to \infty$, both $f(x) \to -\infty$ and $g(x) \to -\infty$.

4. On the same set of axes, sketch the functions $f(x) = \log_2(x)$ and $g(x) = \log_2(x^3)$.



a. Describe a transformation that takes the graph of f to the graph of g.

The graph of g is a vertical scaling of the graph of f by a factor of 3.

b. Use properties of logarithms to justify your observations in part (a).

Using properties of logarithms, we know that $g(x) = \log_2(x^3) = 3\log_2(x) = 3f(x)$. Thus, the graph of f is a vertical scaling of the graph of g by a factor of 3.

5. On the same set of axes, sketch the functions $f(x) = \log_2(x)$ and $g(x) = \log_2\left(\frac{x}{4}\right)$.

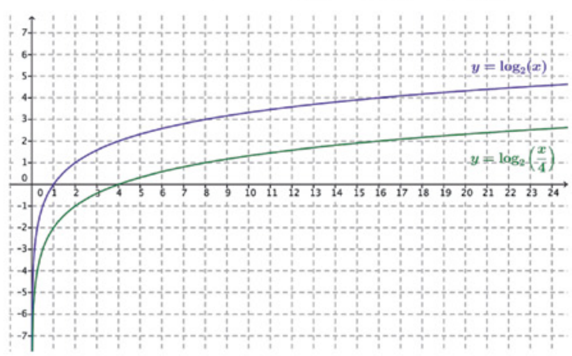


Figure 4.2 (Continued)

a. **Describe a transformation that takes the graph of f to the graph of g.**

The graph of g is the graph of f translated down by 2 units.

b. **Use properties of logarithms to justify your observations in part (a).**

Using properties of logarithms, $g(x) = \log_2\left(\frac{x}{4}\right) = \log_2(x) - \log_2(4) = f(x) - 2$. Thus, the graph of g is a translation of the graph of f down 2 units.

6. **On the same set of axes, sketch the functions $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = \log_2\left(\frac{1}{x}\right)$.**

a. **Describe a transformation that takes the graph of f to the graph of g.**

These two graphs coincide, so the identity transformation takes the graph of f to the graph of g.

b. **Use properties of logarithms to justify your observations in part (a).**

If $\log_{\frac{1}{2}}(x) = y$, then $\left(\frac{1}{2}\right)^y = x$, so $\frac{1}{x} = 2^y$. Then, $y = \log_2$, so $\log_2\left(\frac{1}{x}\right) = \log_{\frac{1}{2}}(x)$; thus, $g(x) = f(x)$ for all $x > 0$.

7. **The figure below shows graphs of the functions $f(x) = \log_3(x)$, $g(x) = \log_5(x)$, and $h(x) = \log_{11}(x)$.**

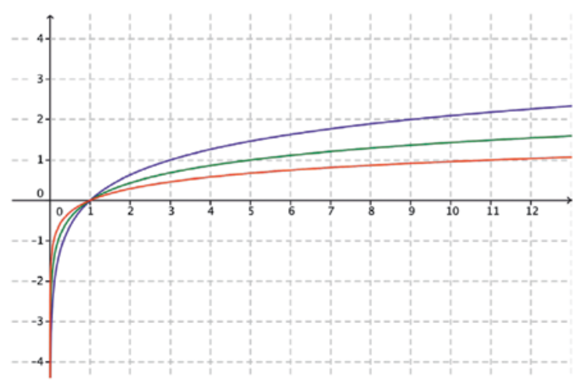
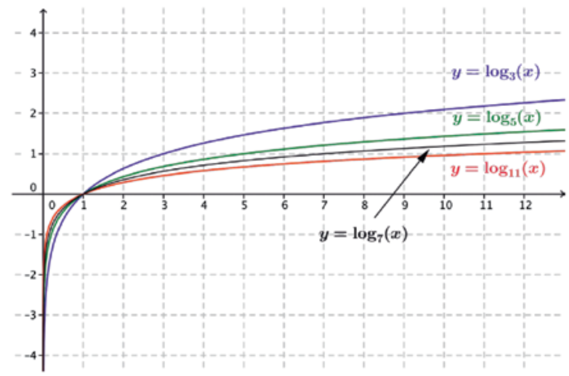


Figure 4.2 (Continued)

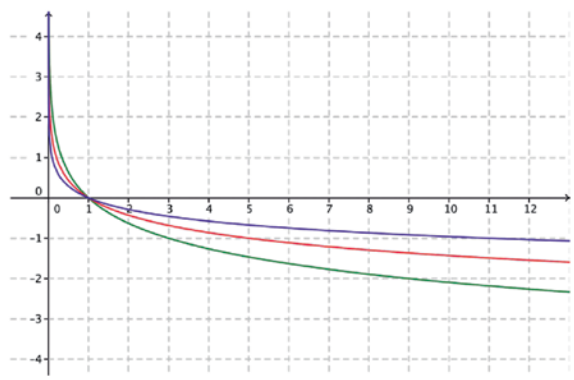
a. **Identify which graph corresponds to which function. Explain how you know.**

The top graph (in blue) is the graph of $f(x) = \log_3(x)$, the middle graph (in green) is the graph of $g(x) = \log_5(x)$, and the lower graph (in red) is the graph of $h(x) = \log_{11}(x)$. We know this because the blue graph passes through the point $(3, 1)$, the green graph passes through the point $(5, 1)$, and the red graph passes through the point $(11, 1)$. We also know that the higher the value of the base b, the flatter the graph, so the graph of the function with the largest base, 11, must be the red graph on the bottom, and the graph of the function with the smallest base, 3, must be the blue graph on the top.

b. **Sketch the graph of $k(x) = \log_7(x)$ on the same axes.**



8. **The figure below shows graphs of the functions $f(x) = \log_{\frac{1}{3}}(x)$, $g(x) = \log_{\frac{1}{5}}(x)$, and $h(x) = \log_{\frac{1}{11}}(x)$.**

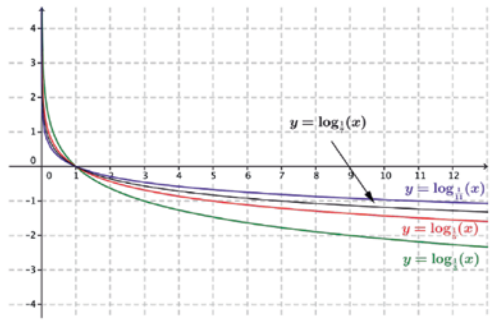


a. **Identify which graph corresponds to which function. Explain how you know.**

The top graph (in blue) is the graph of $h(x) = \log_{\frac{1}{11}}(x)$, the middle graph (in red) is the graph of $g(x) = \log_{\frac{1}{5}}(x)$, and the lower graph is the graph of $f(x) = \log_{\frac{1}{3}}(x)$. We know this because the blue graph passes through the point $(11, -1)$, the red graph passes through the point $(5, -1)$, and the green graph passes through the point $(3, -1)$.

Figure 4.2 (Continued)

b. Sketch the graph of $k(x) = \log_{\frac{1}{7}}(x)$ on the same axes.



9. For each function f, find a formula for the function h in terms of x. Part (a) has been done for you.

a. If $f(x) = x^2 + x$, find $h(x) = f(x + 1)$.

$$\begin{aligned} h(x) &= f(x+1) \\ &= (x+1)^2 + (x+1) \\ &= x^2 + 3x + 2 \end{aligned}$$

b. If $f(x) = \sqrt{x^2 + \frac{1}{4}}$, find $h(x) = f\left(\frac{1}{2}x\right)$.

$h(x) = \frac{1}{2}\sqrt{x^2 + 1}$

c. If $f(x) = \log(x)$, find $h(x) = f\left(\sqrt[3]{10x}\right)$ when $x > 0$.

$h(x) = \frac{1}{3} + \frac{1}{3}\log(x)$

d. If $f(x) = 3^x$, find $h(x) = f(\log_3(x^2 + 3))$.

$h(x) = x^2 + 3$

e. If $f(x) = x^3$, find $h(x) = f\left(\frac{1}{x^3}\right)$ when $x \neq 0$.

$h(x) = \frac{1}{x^6}$

f. If $f(x) = x^3$, find $h(x) = f(\sqrt[3]{x})$.

$h(x) = x$

g. If $f(x) = \sin(x)$, find $h(x) = f\left(x + \frac{\pi}{2}\right)$.

$h(x) = \sin\left(x + \frac{\pi}{2}\right)$

h. If $f(x) = x^2 + 2x + 2$, find $h(x) = f(\cos(x))$.

$h(x) = (\cos(x))^2 + 2\cos(x) + 2$

Figure 4.2 (Continued)

10. For each of the functions f and g below, write an expression for (i) $f(g(x))$, (ii) $g(f(x))$, and (iii) $f(f(x))$ in terms of x. Part (a) has been done for you.

a. $f(x) = x^2$, $g(x) = x + 1$

i. $f(g(x)) = f(x+1)$
$= (x+1)^2$

ii. $g(f(x)) = g(x^2)$
$= x^2 + 1$

iii. $f(f(x)) = f(x^2)$
$= (x^2)^2$
$= x^4$

b. $f(x) = \frac{1}{4}x - 8$, $g(x) = 4x + 1$

i. $f(g(x)) = x - \frac{31}{4}$

ii. $g(f(x)) = x - 31$

iii. $f(f(x)) = \frac{1}{16}x - 10$

c. $f(x) = \sqrt[3]{x+1}$, $g(x) = x^3 - 1$

i. $f(g(x)) = x$

ii. $g(f(x)) = x$

iii. $f(f(x)) = \sqrt[3]{\sqrt[3]{x+1}+1}$

d. $f(x) = x^3$, $g(x) = \frac{1}{x}$

i. $f(g(x)) = \frac{1}{x^3}$

ii. $g(f(x)) = \frac{1}{x^3}$

iii. $f(f(x)) = x^9$

e. $f(x) = |x|$, $g(x) = x^2$

i. $f(g(x)) = |x^2| = x^2$

ii. $g(f(x)) = (|x|)^2 = x^2$

iii. $f(f(x)) = |x|$

Figure 4.2 (Continued)

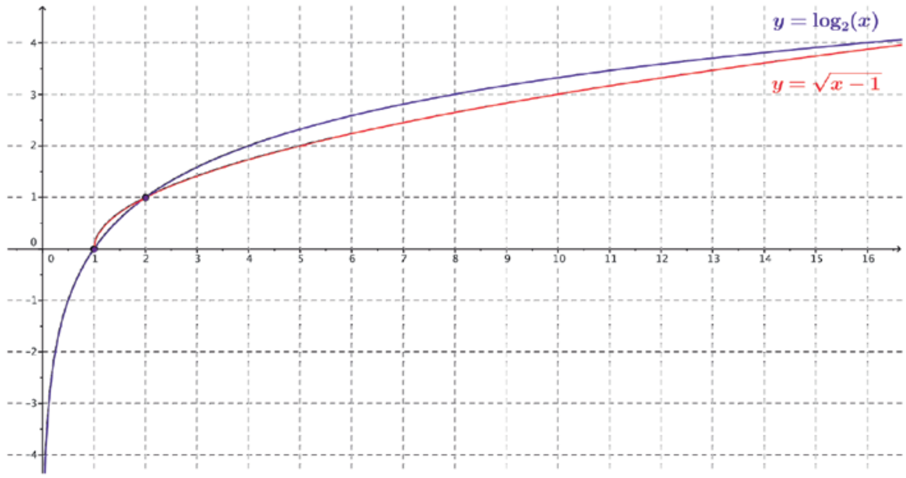
A STORY OF FUNCTIONS Lesson 17 M3

ALGEBRA II

Extension:

11. Consider the functions $f(x) = \log_2(x)$ and $(x) = \sqrt{x-1}$.

a. Use a calculator or other graphing utility to produce graphs of $f(x) = \log_2(x)$ and $g(x) = \sqrt{x-1}$ for $x \leq 17$.



b. Compare the graph of the function $f(x) = \log_2(x)$ with the graph of the function $g(x) = \sqrt{x-1}$. Describe the similarities and differences between the graphs.

They are not the same, but they have a similar shape when $x \geq 1$. Both graphs pass through the points $(1, 0)$ and $(2, 1)$. Both functions appear to approach infinity slowly as $x \to \infty$.

The graph of $f(x) = \log_2(x)$ lies below the graph of $g(x) = \sqrt{x-1}$ on the interval $(1, 2)$, and the graph of f appears to lie above the graph of g on the interval $(2, \infty)$. The logarithm function f is defined for $x > 0$, and the radical function g is defined for $x \geq 1$. Both functions appear to slowly approach infinity as $x \to \infty$.

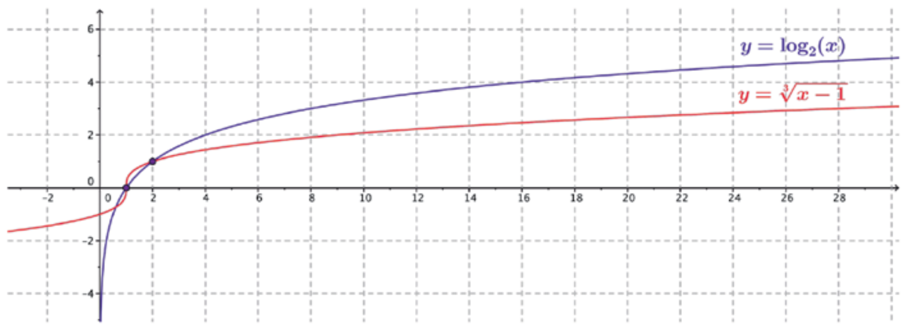
c. Is it always the case that $\log_2(x) > \sqrt{x-1}$ for $x > 2$?

No, for $2 < x \leq 19$, $\log_2(x) > \sqrt{x-1}$. Between 19 and 20, the graphs cross again, and we have $\sqrt{x-1} > \log_2(x)$ for $x \geq 20$.

Figure 4.2 (Continued)

12. Consider the functions $f(x) = \log_2(x)$ and $(x) = \sqrt[3]{x-1}$.

a. Use a calculator or other graphing utility to produce graphs of $f(x) = \log_2(x)$ and $h(x) = \sqrt[3]{x-1}$ for $x \leq 28$.



b. Compare the graph of the function $f(x) = \log_2(x)$ with the graph of the function $h(x) = \sqrt[3]{x-1}$. Describe the similarities and differences between the graphs.

They are not the same, but they have a similar shape when $x \geq 1$. Both graphs pass through the points $(1, 0)$ and $(2, 1)$. Both functions appear to approach infinity slowly as $x \to \infty$.

The graph of $f(x) = \log_2(x)$ lies below the graph of $h(x) = \sqrt[3]{x-1}$ on the interval $(1, 2)$, and the graph of f appears to lie above the graph of h on the interval $(2, \infty)$. The logarithm function f is defined for $x > 0$, and the radical function h is defined for all real numbers x. Both functions appear to approach infinity slowly as $x \to \infty$.

c. Is it always the case that $\log_2(x) > \sqrt[3]{x-1}$ for $x > 2$?

No, if we extend the viewing window on the calculator, we see that the graphs cross again between 983 and 984. Thus, $\log_2(x) > \sqrt[3]{x-1}$ for $2 < x \leq 983$, and $\log_2(x) < \sqrt[3]{x-1}$ for $x \geq 984$.

Figure 4.2 (Continued)

APPROACH TO ASSESSMENT

Assessments provide an opportunity for students to show their learning accomplishments in addition to offering them a pathway to monitor their progress, celebrate successes, examine mistakes, uncover misconceptions, and engage in self-reflection and analysis. A central goal of the assessment system as a whole is to make students aware of their strengths and weaknesses and to give them opportunities to try again, improve, and, in doing so, enjoy the experience of seeing their intelligent, hard work pay off as their skill and understanding increase. Furthermore, the data collected as a result of the assessments represent an invaluable tool in the hands of teachers and provide them with specific information about student understanding to direct their instruction.

In *A Story of Functions*, assessment becomes a regular part of the class routine in the form of daily, mid-module, and end-of-module appraisal. Both the mid-module tasks and the end-of-module tasks are designed to allow for quick scoring that make it possible for teachers to provide instructionally relevant, actionable feedback to students and to monitor resulting student progress to determine the effectiveness of their instruction and make any needed adjustments. These mid-module and end-of-module tasks should be used in combination with instructionally embedded tasks, teacher-developed quizzes, and other formative assessment strategies in order to realize the full benefits of data-driven instruction.

DAILY ASSESSMENTS

Student assessments occur in a variety of ways throughout the module. Questions and sample student responses are provided in the lessons, which can be used to formatively assess students. The Closing at the end of each lesson provides a bullet list of discussion points the teacher can use to consolidate and informally assess the students' understanding of the lesson objectives.

Exit Tickets

Exit tickets are a critical element of the lesson structure. These quick assessments contain specific questions about what was learned that day. The purpose of the exit ticket is twofold: to teach students to grow accustomed to being held individually accountable for the work they have done after one day's instruction and to provide the teacher with valuable evidence of the efficacy of that day's work—which is indispensable for planning purposes.

Problem Sets

Problem Sets give students additional practice on the skills they learn in class each day. The idea is not to introduce brand-new concepts or ideas but to build student confidence with the material learned in class. Having already worked similar problems in class, students work the Problem Set, which gives them a chance to check their understanding at home and confirm that they can do the problems independently.

MID-MODULE ASSESSMENT TASK

A Mid-Module Assessment Task is provided for each module, with the exception of the very short modules. These tasks are specifically tailored to address approximately the first half of the learning objectives for which the module is designed. Careful articulation in a rubric provides guidance in understanding common preconceptions or misconceptions of students for discrete portions of knowledge or skill on their way to proficiency for each standard and preparation for standardized assessments. Typically, these tasks are one class period in length, and students complete them independently. The problems should be new to the students and are not preceded by analogous problems. Teachers may use these tasks either formatively or summatively.

END-OF-MODULE ASSESSMENT TASK

A summative End-of-Module Assessment Task is also administered for each module. These tasks are specifically designed based on the standards addressed in order to gauge students' full range of understanding of the module as a whole and to prepare them for standardized assessments. Some items test understanding of specific standards, while others are synthesis items that assess either understanding of the broader concept addressed in the module or the ability to solve problems by combining knowledge, skills, and understanding. Like the mid-module assessment tasks, these tasks are generally one class period in length and independently completed by the student without assistance. They also should be new to the students and not preceded by analogous problems.

RIGOR IN THE ASSESSMENTS

Each assessment encourages students to demonstrate procedural skill and conceptual understanding. Application problems, including multi-step word problems, are always part of the assessments. Constructed response questions typically involve complex tasks that require students to explain their processes for solving a problem. For these problems, answers alone are insufficient. Students must also be able to thoroughly explain their thought processes. Possible student work may include models, equations, and paragraphs. In any case, the rubrics for these items include elements on judging the thoroughness and correctness of the student's explanation.

CHAPTER 5

Approach to Differentiated Instruction

Teachers are confronted daily with meeting the needs of diverse learners in their classrooms, and differentiating instruction provides a means for meeting this challenge. This chapter explains how A *Story of Functions* integrates Universal Design for Learning (UDL). Marginal notes to the teacher in lessons inform teachers about how to adapt activities to better meet the needs of various student groups.

As noted in the standards (p. 4) “all students must have the opportunity to learn and meet the same high standards if they are to access the knowledge and skills necessary in their post-school lives.” The writers of A *Story of Functions* agree and feel strongly that accommodations cannot be just an extra set of resources for particular students. Instead, scaffolding must be folded into the curriculum in such a way that it is part of its very DNA. Said another way, faithful adherence to the modules is the primary scaffolding tool.

The modules that make up A *Story of Functions* propose that the components of excellent math instruction do not change based on the audience. That said, specific resources within this curriculum highlight strategies that can provide critical access for all students.

Research-based UDL has provided a structure for thinking about how to meet the needs of diverse learners. Broadly, that structure asks teachers to consider multiple means of representation, multiple means of action and expression, and multiple means of engagement. Charts at the end of this section offer suggested scaffolds, using this framework, for English language learners, students with disabilities, students performing above grade level, and students performing below grade level. UDL offers ideal settings for multiple entry points for students and minimizes instructional barriers to learning. Many of the suggestions on the chart are applicable to other students and overlapping populations.

In addition, individual lessons contain marginal notes to teachers (in text boxes) highlighting specific UDL information about scaffolds that might be employed with particular intentionality when working with students. These tips are strategically placed in the lesson where teachers might use the strategy to the best advantage.

It is important to note that the scaffolds and accommodations integrated into A *Story of Functions* might change how learners access information and demonstrate learning; they

do not substantially alter the instructional level, content, or performance criteria. Rather, they provide teachers with choices in how students access content and demonstrate their knowledge and ability.

We encourage teachers to pay particular attention to the manner in which knowledge is sequenced in A *Story of Functions* and to capitalize on that sequence when working with special student populations. Most lessons contain a suggested teaching sequence that moves from simple to complex, starting, for example, with an introductory problem for a math topic and building up inductively to the general case encompassing multifaceted ideas. By breaking down problems from simple to complex, teachers can locate specific steps that students are struggling with or stretch out problems for students who desire a challenge.

Another vitally important component for meeting the needs of all students is the constant flow of data from student work. A *Story of Functions* provides daily tracking through exit tickets for each lesson, as well as Mid- and End-of-Module Assessment tasks to determine student understanding at benchmark points. These tasks should accompany teacher-made test items in a comprehensive assessment plan. Such data flow keeps teaching practice firmly grounded in student learning and makes incremental forward movement possible. A culture of precise error correction in the classroom breeds a comfort with data that is nonpunitive and honest. When feedback is provided with emotional neutrality, students understand that making mistakes is part of the learning process: "Students, for the next five minutes, I will be meeting with Brenda, Mehmet, and Jeremy. On Problem 7, they did not remember to *use already proven facts when writing the proof*." Conducting such sessions then provides the teacher the opportunity to quickly assess if students need to start at a simpler level or just need more monitored practice now that their eyes have been opened to their mistakes.

Good mathematics instruction, like any successful coaching, involves demonstration, modeling, and a lot of intelligent practice. In math, just as in sports, skill is acquired incrementally; as the student acquires greater skill, more complexity is added to the work, and the student's proficiency grows. The careful sequencing of the mathematics and the many scaffolds that have been designed into A *Story of Functions* make it an excellent curriculum for meeting the needs of all students, including those with special and unique learning modes.

SCAFFOLDS FOR ENGLISH LANGUAGE LEARNERS

English language learners (ELLs) provide a variety of experiences that can add to the classroom environment. Their differences do not translate directly to shortfalls in knowledge base but rather present an opportunity to enrich the teaching and learning. The following chart provides a bank of suggestions within the UDL framework to aid ELLs. Variations on these accommodations are elaborated within lessons, demonstrating how and when they might be used.

Provide Multiple Means of Representation	• Introduce essential terms and vocabulary prior to the mathematics instruction. • Clarify, compare, and make connections to math words in discussion, particularly during and after practice. • Highlight critical vocabulary in discussion. • Let students use models and gestures to calculate and explain. For example, a student searching to define *translation* draws a picture of a triangle slide in the coordinate plane. • Teach students how to ask questions (such as, "Do you agree?" and "Why do you think so?") to extend think-pair-share conversations. Model and post conversation starters, such as, "I agree because …" "Can you explain how you solved it?" "I noticed that …" "Your solution is different from [the same as] mine because …" "My mistake was to …" • Connect language (such as *tens*) with concrete and pictorial experiences.
Provide Multiple Means of Action and Expression	• Know, use, and make the most of student cultural and home experiences. Build on the student's background knowledge. • Check for understanding frequently (e.g., "Show me what you are thinking.") to benefit those who may shy away from asking questions. • Couple teacher-talk with illustrative gestures. Vary your voice to guide comprehension. Speak dynamically with expression. Make eye-to-eye contact, and speak slowly and distinctly. • Vary the grouping in the classroom, such as sometimes using small group instruction to help ELLs learn to negotiate vocabulary with classmates and other times using native language support to allow a student to find full proficiency of the mathematics first. • Provide sufficient wait time to allow the students to process the meanings in the different languages. • Listen intently in order to uncover the math content in the student's speech. • Keep teacher-talk clear and concise. • Point to visuals while speaking, using your hands to clearly indicate the image that corresponds to your words. • Get students up and moving, coupling language with motion. • Celebrate improvement. Intentionally highlight student math success frequently.
Provide Multiple Means of Engagement	• Provide a variety of ways to respond: oral, choral, student boards, concrete models, pictorial models, pair share, small group share. • Treat everyday and first language experiences as resources, not as obstacles. Be aware of translations, such as *denominator* in English and *denominador* in Spanish. • Provide oral options for assessment rather than multiple choice. • Cultivate a math discourse of synthesis, analysis, and evaluation rather than simplified language. • Support oral or written response with sentence frames, such as "In the ordered pair ______, _____ is the *x*-coordinate and _____ is the *y*-coordinate." • Ask questions to probe what students mean as they attempt expression in a second language. • Scaffold questioning to guide connections, analysis, and mastery. • Let students choose the language they prefer for arithmetic computation and discourse.

SCAFFOLDS FOR STUDENTS WITH DISABILITIES

Individualized education programs (IEPs) or Section 504 accommodation plans should be the first source of information for designing instruction for students with disabilities. The following chart provides an additional bank of suggestions within the UDL framework for strategies to use with these students in your class. Variations on these scaffolds are elaborated at particular points within lessons with text boxes at appropriate points, demonstrating how and when they might be used.

Provide Multiple Means of Representation	• Teach from simple to complex at the student's pace. • Clarify, compare, and make connections to math words in discussion, particularly during and after practice. • Partner key words with visuals (e.g., a photo of *ticket*) and gestures (e.g., for *paid*). Connect language (such as *tens*) with concrete and pictorial experiences. Couple teacher-talk with math-they-can-see, such as models. Let students use models and gestures to calculate and explain. • Teach students how to ask questions (such as, "Do you agree?" and "Why do you think so?") to extend think-pair-share conversations. Model and post conversation starters, such as: "I agree because …" "Can you explain how you solved it?" "I noticed that …" "Your solution is different from [the same as] mine because …" "My mistake was to …" • Enlarge print for visually impaired learners. • Use student boards to work on one calculation at a time. • Invest in or make math picture dictionaries or word walls.
Provide Multiple Means of Action and Expression	• Provide a variety of ways to respond: oral, choral, student boards, concrete models, pictorial models, pair share, and small group share. For example, use student boards to adjust partner share for hearing-impaired students. Partners can jot questions and answers to one another on boards. Use vibrations or visual signs (such as a clap rather than a snap or saying, "Show") to elicit responses from hearing-impaired students. • Vary choral response with written response (number sentences and models) on student boards to ease linguistic barriers. Support oral or written response with sentence frames, such as "In the ordered pair_______, _____ is the *x*-coordinate and _____ is the *y*-coordinate." • Adjust oral fluency games by using student and teacher boards or hand signals. Use visual signals or vibrations to elicit responses. • Adjust wait time for interpreters of hearing-impaired students. • Select numbers and tasks that are just right for learners. • Model each step of the algorithm before students begin. • Give students a few extra minutes to process the information before giving the signal to respond. • Assess by multiple means, including "show and tell" rather than written. • Elaborate on the problem-solving process. Read word problems aloud. Post a visual display of the problem-solving process. Have students check off or highlight each step as they work. Talk through the problem-solving process step-by-step to demonstrate the thinking process. Before students solve, ask questions for comprehension. Teach students to use self-questioning techniques, such as, "Does my answer make sense?" • Concentrate on goals for accomplishment within a time frame as opposed to a task frame. Extend the time for the task. Guide students to evaluate process and practice. Have students ask themselves, "How did I improve? What did I do well?" • Focus on students' mathematical reasoning (i.e., their ability to make comparisons, describe patterns, generalize, explain conclusions, specify claims, and use models), not their accuracy in language.

Provide Multiple Means of Engagement	• Make eye-to-eye contact, and keep teacher-talk clear and concise. Speak clearly when checking answers for problems. • Check frequently for understanding (e.g., *show*). Listen intently in order to uncover the math content in the student's speech. Use nonverbal signals, such as thumbs-up. Assign a buddy or a group to clarify directions or processes. • Teach in small chunks so that students get a lot of practice with one step at a time. • Know, use, and make the most of Deaf culture and sign language. • Use songs, rhymes, or rhythms to help students remember key concepts. • Point to visuals and captions while speaking, using your hands to clearly indicate the image that corresponds to your words. • Incorporate activity. Get students up and moving, coupling language with motion. • Celebrate improvement. Intentionally highlight student math success frequently. • Follow predictable routines to allow students to focus on content rather than behavior. • Allow everyday and first language to express math understanding. • Allow students to lead group and pair-share activities. • Provide learning aids, such as calculators and computers, to help students focus on conceptual understanding.

SCAFFOLDS FOR STUDENTS PERFORMING BELOW GRADE LEVEL

The following chart provides a bank of suggestions within the UDL framework for accommodating students who are below grade level in your class. Variations on these accommodations are elaborated within lessons, demonstrating how and when they might be used.

Provide Multiple Means of Representation	• Model problem-solving sets with drawings and graphic organizers. • Guide students as they select and practice using their own graphic organizers and models to solve. • Use direct instruction for vocabulary with visual or concrete representations. • Use explicit directions with steps and procedures enumerated. Guide students through initial practice promoting gradual independence. • Use alternative methods of delivery of instruction, such as recordings and videos, that can be accessed independently or repeated if necessary. • Scaffold complex concepts, and provide leveled problems for multiple entry points.
Provide Multiple Means of Action and Expression	• First use manipulatives or real objects when appropriate; then make the transition from concrete to pictorial to abstract. • Have students restate their learning for the day. Ask for a different representation in the restatement: "Would you restate that answer in a different way or show me by using a diagram?" • Encourage students to explain their thinking and strategy for the solution. • Choose numbers and tasks that are just right for learners but teach the same concepts. • Adjust numbers in calculations to suit the learner's level.

Provide Multiple Means of Engagement	• Clearly model steps, procedures, and questions to ask when solving. • Cultivate peer-assisted learning interventions for instruction (e.g., dictation) and practice, particularly for computation work (e.g., peer modeling). Have students work together to solve and then check their solutions. • Teach students to ask themselves questions as they solve: "Do I know the meaning of all the words in this problem? What is being asked? Do I have all of the information I need? What do I do first? What is the order to solve this problem? What calculations do I need to make?" • Practice routine to ensure smooth transitions. • Set goals with the students regarding next steps and what to focus on next.

SCAFFOLDS FOR STUDENTS PERFORMING ABOVE GRADE LEVEL

The following chart provides a bank of suggestions within the UDL framework for accommodating students who are above grade level in your class. Variations on these accommodations are elaborated within lessons, demonstrating how and when they might be used.

Provide Multiple Means of Representation	• Teach students how to ask questions (such as, "Do you agree?" and "Why do you think so?") to extend think-pair-share conversations. Model and post conversation starters, such as, "I agree because …" "Can you explain how you solved it?" "I noticed that …" "Your solution is different from [the same as] mine because …" "My mistake was to …" • Incorporate written reflection, evaluation, and synthesis. • Allow creativity in expression and modeling solutions.
Provide Multiple Means of Action and Expression	• Encourage students to explain their reasoning both orally and in writing. • Extend exploration of math topics and offer choices of independent or group assignments for early finishers. • Encourage students to notice and explore patterns and to identify rules and relationships in math. Have students share their observations in discussion and writing (e.g., journaling). • Foster their curiosity about numbers and mathematical ideas. Facilitate research and exploration through discussion, experiments, Internet searches, trips, and other means. • Let students choose their mode of response: written, oral, concrete, pictorial, or abstract. • Increase the pace. Offer two word problems to solve rather than one. Adjust the difficulty level by increasing the number of steps (e.g., change a one-step problem to a two-step problem), increasing numbers to millions or decreasing numbers to decimals or fractions. • Let students compose their own word problems to show their mastery and extension of the content.
Provide Multiple Means of Engagement	• Push student comprehension into higher levels of Bloom's Taxonomy with questions such as, "What would happen if …?" "Can you propose an alternative?" "How would you evaluate …?" "What choice would you have made?" Ask "why?" and "what if?" questions. • Accept and elicit student ideas and suggestions for ways to extend games. • Cultivate student persistence in problem solving, and do not overlook their need for guidance and support.

CHAPTER 6

Course Module Summary and Unpacking of Standards

Although the standards delineate what students should learn, teachers must still fill in the blanks when it comes to translating this information for use in their daily classroom practice. Questions often remain about how concepts should be connected and where parameters should be set. The content of each course in *A Story of Functions* is contained in four to five modules that span an academic year.

This chapter presents key information from the modules to provide an overview of the content and explain the mathematical progression. The standards are translated for teachers and a fuller picture is drawn of the teaching and learning that should take place through the school year. Each module comprises the following:

An Overview of the module

Focus Standards

Extension Standards (when applicable)

Foundational Standards

Focus Standards for Mathematical Practice

Topic Overview narratives and Student Outcomes for each lesson

The Module Overview provides a broad overview for the entire module and includes information on the sequencing of the topics. The Topic Overviews unpack the standards by providing additional details about the skills and concepts students should master at the lesson level. Further unpacking of the standards continues as the Student Outcomes illuminate what students should know and be able to do to demonstrate mastery. This information, taken together for all modules, tells a story of the mathematics students should learn for the course.

MODULE 1: POLYNOMIAL, RATIONAL, AND RADICAL RELATIONSHIPS

OVERVIEW

In this module, students draw on their foundation of the analogies between polynomial arithmetic and base-10 computation, focusing on properties of operations, particularly the distributive property (A-SSE.B.2, A-APR.A.1). Students connect multiplication of polynomials with multiplication of multi-digit integers and division of polynomials with long division of integers (A-APR.A.1, A-APR.D.6). Students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations (A-APR.B.3). Students explore the role of factoring, as both an aid to the algebra and to the graphing of polynomials (A-SSE.2, A-APR.B.2, A-APR.B.3, F-IF.C.7c). Students continue to build on the reasoning process of solving equations as they solve polynomial, rational, and radical equations, as well as linear and nonlinear systems of equations (A-REI.A.1, A-REI.A.2, A-REI.C.6, A-REI.C.7). The module culminates with the fundamental theorem of algebra as the ultimate result in factoring. Students pursue connections to applications in prime numbers in encryption theory, Pythagorean triples, and modeling problems.

An additional theme of this module is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers. Students use appropriate tools to analyze the key features of a graph or table of a polynomial function and relate those features back to the two quantities that the function is modeling in the problem (F-IF.C.7c).

The module comprises 40 lessons; 5 days are reserved for administering the Mid- and End-of-Module Assessments, returning the assessments, and remediating or providing further applications of the concepts. The Mid-Module Assessment follows Topic B. The End-of-Module Assessment follows Topic D.

FOCUS STANDARDS

Reason quantitatively and use units to solve problems.

N-Q.A.2[1] Define appropriate quantities for the purpose of descriptive modeling.★

Perform arithmetic operations with complex numbers.

N-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

N-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

Interpret the structure of expressions.

A-SSE.A.2[2] Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

Understand the relationship between zeros and factors of polynomials.

A-APR.B.2[3] Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

A-APR.B.3[4] Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

A-APR.C.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*

Rewrite rational expressions.

A-APR.D.6[5] Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1[6] Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

A-REI.A.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable.

A-REI.B.4[7] Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations.

A-REI.C.6[8] Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

A-REI.C.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ *and the circle* $x^2 + y^2 = 3$.

Analyze functions using different representations.

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph (by hand in simple cases and using technology for more complicated cases).★

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Translate between the geometric description and the equation for a conic section.

G-GPE.A.2 Derive the equation of a parabola given a focus and directrix.

EXTENSION STANDARDS

The (+) standards that follow are provided as an extension to Module 1 of the Algebra II course to provide coherence to the curriculum. They are used to introduce themes and concepts that are fully covered in the Precalculus course.

Use complex numbers in polynomial identities and equations.

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.

N-CN.C.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Rewrite rational expressions.

A-APR.C.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

FOUNDATIONAL STANDARDS

Use properties of rational and irrational numbers.

N-RN.B.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Reason quantitatively and use units to solve problems.

N-Q.A.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.★

Interpret the structure of expressions.

A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of* P *and a factor not depending on* P.

Write expressions in equivalent forms to solve problems.

A-SSE.B.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

Perform arithmetic operations on polynomials.

A-APR.A.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Create equations that describe numbers or relationships.

A-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

A-CED.A.3 Represent constraints by equations or inequalities and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*★

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning used in solving equations. *For example, rearrange Ohm's law* V = IR *to highlight resistance* R.★

Solve equations and inequalities in one variable.

A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

A-REI.B.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

Solve systems of equations.

A-REI.C.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

Represent and solve equations and inequalities graphically.

A-REI.D.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

A-REI.D.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

Translate between the geometric description and the equation for a conic section.

G-GPE.A.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

FOCUS STANDARDS FOR MATHEMATICAL PRACTICE

MP.1 *Make sense of problems and persevere in solving them.* Students discover the value of equating factored terms of a polynomial to zero as a means of solving equations involving polynomials. Students solve rational equations and simple radical equations, while considering the possibility of extraneous solutions and verifying each solution before drawing conclusions about the problem. Students solve systems of linear equations and linear and quadratic pairs in two variables. Further, students come to understand that the complex number system provides solutions to the equation $x^2 + 1 = 0$ and higher-degree equations.

MP.2 *Reason abstractly and quantitatively.* Students apply polynomial identities to detect prime numbers and discover Pythagorean triples. Students also learn to make sense of remainders in polynomial long division problems.

MP.4 *Model with mathematics.* Students use primes to model encryption. Students transition between verbal, numerical, algebraic, and graphical thinking in analyzing applied polynomial problems. Students model a cross-section of a riverbed with a polynomial, estimate fluid flow with their algebraic model, and fit polynomials to data. Students model the locus of points at equal distance between a point (focus) and a line (directrix), discovering the parabola.

MP.7 *Look for and make use of structure.* Students connect long division of polynomials with the long-division algorithm of arithmetic and perform polynomial division in an abstract setting to derive the standard polynomial identities. Students recognize structure in the graphs of polynomials in factored form and develop refined techniques for graphing. Students discern the structure of rational expressions by comparing to analogous arithmetic problems. Students perform geometric operations on parabolas to discover congruence and similarity.

MP.8 *Look for and express regularity in repeated reasoning.* Students understand that polynomials form a system analogous to the integers. Students apply polynomial identities to detect prime numbers and discover Pythagorean triples. Students recognize factors of expressions and develop factoring techniques. Further, students understand that all quadratics can be written as a product of linear factors in the complex realm.

MODULE TOPIC SUMMARIES

Topic A: Polynomials—From Base Ten to Base X

In Topic A, students draw on their foundation of the analogies between polynomial arithmetic and base-10 computation, focusing on properties of operations, particularly the distributive property. In Lesson 1, students write polynomial expressions for sequences by examining successive differences. They are engaged in a lively lesson that emphasizes thinking and reasoning about numbers and patterns and equations. In Lesson 2, students use a variation of the area model referred to as the tabular method to represent polynomial multiplication and connect that method back to application of the distributive property.

In Lesson 3, students continue using the tabular method and analogies to the system of integers to explore division of polynomials as a missing factor problem. In this lesson, students also take time to reflect on and arrive at generalizations for questions such as how to predict the degree of the resulting sum when adding two polynomials. In Lesson 4, students are ready to ask and answer whether long division can work with polynomials too and how it compares with the tabular method of finding the missing factor. Lesson 5 gives students additional practice on all operations with polynomials and offers an opportunity to

examine the structure of expressions, such as recognizing that $\frac{n(n+1)(2n+1)}{6}$ is a third degree polynomial expression with leading coefficient $\frac{1}{3}$ without having to expand it out.

In Lesson 6, students extend their facility with dividing polynomials by exploring a more generic case; rather than dividing by a factor such as $(x+3)$, they divide by the factor $(x+a)$ or $(x-a)$. This gives them the opportunity to discover the structure of special products such as $(x-a)(x^2+ax+a^2)$ in Lesson 7 and go on to use those products in Lessons 8–10 to employ the power of algebra over the calculator. In Lesson 8, they find they can use special products to uncover mental math strategies and answer questions such as whether or not $2^{100}-1$ is prime. In Lesson 9, they consider how these properties apply to expressions that contain square roots. Then, in Lesson 10, they use special products to find Pythagorean triples.

The topic culminates with Lesson 11 and the recognition of the benefits of factoring and the special role of zero as a means for solving polynomial equations.

Focus Standards:	A-SSE.A.2	Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.
	A-APR.C.4	Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
Instructional Days:	11	

Student Outcomes

Lesson 1: Successive Differences in Polynomials

- Students write explicit polynomial expressions for sequences by investigating successive differences of those sequences.

Lesson 2: The Multiplication of Polynomials

- Students develop the distributive property for application to polynomial multiplication. Students connect multiplication of polynomials with multiplication of multi-digit integers.

Lesson 3: The Division of Polynomials

- Students develop a division algorithm for polynomials by recognizing that division is the inverse operation of multiplication.

Lesson 4: Comparing Methods—Long Division, Again?

- Students connect long division of polynomials with the long division algorithm of arithmetic and use this algorithm to rewrite rational expressions that divide without a remainder.

Lesson 5: Putting It All Together

- Students perform arithmetic operations on polynomials and write them in standard form.
- Students understand the structure of polynomial expressions by quickly determining the first and last terms if the polynomial were to be written in standard form.

Lesson 6: Dividing by $x - a$ and by $x + a$

- Students work with polynomials with constant coefficients to derive and use polynomial identities.

Lesson 7: Mental Math

- Students perform arithmetic by using polynomial identities to describe numerical relationships.

Lesson 8: The Power of Algebra—Finding Primes

- Students apply polynomial identities to the detection of prime numbers.

Lesson 9: Radicals and Conjugates

- Students understand that the sum of two square roots (or two cube roots) is not equal to the square root (or cube root) of their sum.
- Students convert expressions to simplest radical form.
- Students understand that the product of conjugate radicals can be viewed as the difference of two squares.

Lesson 10: The Power of Algebra—Finding Pythagorean Triples

- Students explore the difference of two squares identity $x^2 - y^2 = (x - y)(x + y)$ in the context of finding Pythagorean triples.

Lesson 11: The Special Role of Zero in Factoring

- Students find solutions to polynomial equations where the polynomial expression is not factored into linear factors.
- Students construct a polynomial function that has a specified set of zeros with stated multiplicity.

Topic B: Factoring—Its Use and Its Obstacles

Armed with a newfound knowledge of the value of factoring, students develop their facility with factoring and then apply the benefits to graphing polynomial equations in Topic B. In Lessons 12–13, students are presented with the first obstacle to solving equations successfully. Whereas dividing a polynomial by a given factor to find a missing factor is easily accessible, factoring without knowing one of the factors is challenging. Students recall the work with factoring done in Algebra I and expand on it to master factoring polynomials with degree greater than two, emphasizing the technique of factoring by grouping.

In Lessons 14–15, students find that another advantage to rewriting polynomial expressions in factored form is how easily a polynomial function written in this form can be graphed. Students read word problems to answer polynomial questions by examining key features of their graphs. They notice the relationship between the number of times a factor is repeated and the behavior of the graph at that zero (i.e., when a factor is repeated an even number of times, the graph of the polynomial touches the x-axis and "bounces" back off, whereas when a factor occurs only once or an odd number of times, the graph of the polynomial at that zero "cuts through" the x-axis). In these lessons, students compare hand plots to graphing-calculator plots and zoom in on the graph to examine its features more closely.

In Lessons 16–17, students encounter a series of more serious modeling questions associated with polynomials, developing their fluency in translating between verbal, numeric, algebraic, and graphical thinking. One example of the modeling questions posed in this lesson is how to find the maximum possible volume of a box created from a flat piece of cardboard with fixed dimensions.

In Lessons 18–19, students are presented with their second obstacle: "What if there is a remainder?" They learn the remainder theorem and apply it to further understand the connection between the factors and zeros of a polynomial and how this relates to the graph of a polynomial function. Students explore how to determine the smallest possible degree for a depicted polynomial and how information such as the value of the y-intercept is reflected in the equation of the polynomial.

The topic culminates with two modeling lessons (Lessons 20–21) involving approximating the area of the cross-section of a riverbed to model the volume of flow. The problem description includes a graph of a polynomial equation that could be used to model the situation, and students are challenged to find the polynomial equation itself.

Focus Standards:	N-Q.A.2	Define appropriate quantities for the purpose of descriptive modeling.*
	A-SSE.A.2	Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.
	A-APR.B.2	Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
	A-APR.B.3	Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.
	A-APR.D.6	Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
	F-IF.C.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.* c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.
Instructional Days:	10	

Student Outcomes

Lesson 12: Overcoming Obstacles in Factoring

- Students factor certain forms of polynomial expressions by using the structure of the polynomials.

Lesson 13: Mastering Factoring

- Students use the structure of polynomials to identify factors.

Lesson 14: Graphing Factored Polynomials

- Students use the factored forms of polynomials to find zeros of a function.
- Students use the factored forms of polynomials to sketch the components of graphs between zeros.

Lesson 15: Structure in Graphs of Polynomial Functions

- Students graph polynomial functions and describe end behavior based on the degree of the polynomial.

Lesson 16: Modeling with Polynomials—An Introduction

- Students transition between verbal, numerical, algebraic, and graphical thinking in analyzing applied polynomial problems.

Lesson 17: Modeling with Polynomials—An Introduction

- Students interpret and represent relationships between two types of quantities with polynomial functions.

Lesson 18: Overcoming a Second Obstacle in Factoring—What If There Is a Remainder?

- Students rewrite simple rational expressions in different forms, including representing remainders when dividing.

Lesson 19: The Remainder Theorem

- Students know and apply the remainder theorem and understand the role zeros play in the theorem.

Lesson 20: Modeling Riverbeds with Polynomials

- Students learn to fit polynomial functions to data values.

Lesson 21: Modeling Riverbeds with Polynomials

- Students model a cross-section of a riverbed with a polynomial function and estimate fluid flow with their algebraic model.

Topic C: Solving and Applying Equations—Polynomial, Rational, and Radical

In Topic C, students continue to build on the reasoning used to solve equations and their fluency in factoring polynomial expressions. In Lesson 22, students expand their understanding of the division of polynomial expressions to rewriting simple rational expressions (A-APR.D.6) in equivalent forms. In Lesson 23, students learn techniques for comparing rational expressions numerically, graphically, and algebraically. In Lessons 24–25, students learn to rewrite simple rational expressions by multiplying, dividing, adding, or subtracting two or more expressions. They begin to connect operations with rational numbers to operations on rational expressions. The practice of rewriting rational expressions in equivalent forms in Lessons 22–25 is carried over to solving rational equations in Lessons 26 and 27. Lesson 27 also includes working with word problems that require the use of rational equations. Lessons 28–29 turn to radical equations. Students learn to look for extraneous solutions to these equations as they did for rational equations.

In Lessons 30–32, students solve and graph systems of equations including systems of one linear equation and one quadratic equation and systems of two quadratic equations. Next, in Lessons 33–35, students study the definition of a parabola as they first learn to derive the equation of a parabola given a focus and a directrix and later to create the equation of the parabola in vertex form from the coordinates of the vertex and the location of either the focus or directrix. Students build on their understanding of rotations and translations from Geometry as they learn that any given parabola is congruent to the one given by the equation $y = ax^2$ for some value of a, and that all parabolas are similar.

Focus Standards:	A-APR.D.6	Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
	A-REI.A.1	Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
	A-REI.A.2	Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

	A-REI.B.4	Solve quadratic equations in one variable. b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.
	A-REI.C.6	Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
	A-REI.C.7	Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ *and the circle* $x^2 + y^2 = 3$.
	G-GPE.A.2	Derive the equation of a parabola given a focus and directrix.
Instructional Days:	14	

Student Outcomes

Lesson 22: Equivalent Rational Expressions

- Students define rational expressions and write them in equivalent forms.

Lesson 23: Comparing Rational Expressions

- Students compare rational expressions by writing them in different but equivalent forms.

Lesson 24: Multiplying and Dividing Rational Expressions

- Students multiply and divide rational expressions and simplify using equivalent expressions.

Lesson 25: Adding and Subtracting Rational Expressions

- Students perform addition and subtraction of rational expressions.

Lesson 26: Solving Rational Equations

- Students solve rational equations, monitoring for the creation of extraneous solutions.

Lesson 27: Word Problems Leading to Rational Equations

- Students solve word problems using models that involve rational expressions.

Lesson 28: A Focus on Square Roots

- Students solve simple radical equations and understand the possibility of extraneous solutions. They understand that care must be taken with the role of square roots so as to avoid apparent paradoxes.
- Students explain and justify the steps taken in solving simple radical equations.

Lesson 29: Solving Radical Equations

- Students develop facility in solving radical equations.

Lesson 30: Linear Systems in Three Variables

- Students solve linear systems in three variables algebraically.

Lesson 31: Systems of Equations

- Students solve systems of linear equations in two variables and systems of a linear and a quadratic equation in two variables.
- Students understand that the points at which the two graphs of the equations intersect correspond to the solutions of the system.

Lesson 32: Graphing Systems of Equations

- Students develop facility with graphical interpretations of systems of equations and the meaning of their solutions on those graphs. For example, they can use the distance formula to find the distance between the centers of two circles and thereby determine whether the circles intersect in 0, 1, or 2 points.
- By completing the squares, students can convert the equation of a circle in general form to the center-radius form and thus find the radius and center. They can also convert the center-radius form to the general form by removing parentheses and combining like terms.
- Students understand how to solve and graph a system consisting of two quadratic equations in two variables.

Lesson 33: The Definition of a Parabola

- Students model the locus of points at equal distance between a point (focus) and a line (directrix). They construct a parabola and understand this geometric definition of the curve. They use algebraic techniques to derive the analytic equation of the parabola.

Lesson 34: Are All Parabolas Congruent?

- Students learn the vertex form of the equation of a parabola and how it arises from the definition of a parabola.
- Students perform geometric operations, such as rotations, reflections, and translations, on arbitrary parabolas to discover standard representations for their congruence classes. In doing so, they learn that all parabolas with the same distance p between the focus and the directrix are congruent to the graph of $y = \frac{1}{2p}x^2$.

Lesson 35: Are All Parabolas Similar?

- Students apply the geometric transformation of dilation to show that all parabolas are similar.

Topic D: A Surprise from Geometry—Complex Numbers Overcome All Obstacles

In Topic D, students extend their facility with finding zeros of polynomials to include complex zeros. Lesson 36 presents a third obstacle to using factors of polynomials to solve polynomial equations. Students begin by solving systems of linear and nonlinear equations to which no real solutions exist and then relate this to the possibility of quadratic equations with no real solutions. Lesson 37 introduces complex numbers through their relationship to geometric transformations. That is, students observe that scaling all numbers on a number line by a factor of –1 turns the number line out of its one-dimensionality and rotates it 180° through the plane. They then answer the question, “What scale factor could be used to create a rotation of 90°?” In Lesson 38, students discover that complex numbers have real uses; in fact, they can be used in finding real solutions of polynomial equations. In Lesson 39, students develop facility with properties and operations of complex numbers and then apply that facility to factor polynomials with complex zeros. Lesson 40 brings the module to a close with the result that every polynomial can be rewritten as the product of linear factors, which is not possible without complex numbers. Even though standards N-CN.C.8 and N-CN.C.9 are not assessed at the Algebra II level, they are included instructionally to develop further conceptual understanding.

Focus Standards:	N-CN.A.1	Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
	N-CN.A.2	Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
	N-CN.C.7	Solve quadratic equations with real coefficients that have complex solutions.
	A-REI.A.2	Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.
	A-REI.B.4	Solve quadratic equations in one variable. b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a + bi$ for real numbers a and b.
	A-REI.C.7	Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ *and the circle* $x^2 + y^2 = 3$.
Instructional Days:	5	

Student Outcomes

Lesson 36: Overcoming a Third Obstacle to Factoring—What If There Are No Real Number Solutions?

- Students understand the possibility that there might be no real number solution to an equation or system of equations. Students identify these situations and make the appropriate geometric connections.

Lesson 37: A Surprising Boost from Geometry

- Students write a complex number in the form $a+bi$, where a and b are real numbers and the imaginary unit i satisfies $i^2 = -1$. Students geometrically identify i as a multiplicand effecting a 90° counterclockwise rotation of the real number line. Students locate points corresponding to complex numbers in the complex plane.
- Students understand complex numbers as a superset of the real numbers (i.e., a complex $a+bi$ is real when $b = 0$). Students learn that complex numbers share many similar properties of the real numbers: associative, commutative, distributive, addition/subtraction, multiplication, and so on.

Lesson 38: Complex Numbers as Solutions to Equations

- Students solve quadratic equations with real coefficients that have complex solutions (N-CN.C.7). They recognize when the quadratic formula gives complex solutions and write them as $a+bi$ for real numbers a and b (A-REI.B.4b).

Lesson 39: Factoring Extended to the Complex Realm

- Students solve quadratic equations with real coefficients that have complex solutions. Students extend polynomial identities to the complex numbers.
- Students note the difference between solutions to a polynomial equation and the x-intercepts of the graph of that equation.

Lesson 40: Obstacles Resolved—A Surprising Result

- Students understand the fundamental theorem of algebra and that all polynomial expressions factor into linear terms in the realm of complex numbers.

MODULE 2: TRIGONOMETRIC FUNCTIONS

OVERVIEW

Module 2 builds on students' previous work with units (N-Q.A.1) and with functions (F-IF.A.1, F-IF.A.2, F-IF.B.4, F-IF.C.7e, F-BF.A.1, F-BF.B.3) from Algebra I and with trigonometric ratios and circles (G-SRT.C.6, G-SRT.C.7, G-SRT.C.8) from high school Geometry. Included in Topic A is preparation for extension standard F-TF.A.3. Extension standard F-TF.C.9 is also discussed in Topic B as preparation for the Precalculus and Advanced Topics course.

Topic A starts by asking students to graph the height of a passenger car on a Ferris wheel as a function of how much rotation it has undergone and uses that study to help define the sine, cosine, and tangent functions as functions from all (or most) real numbers to the real numbers. A precise definition of sine and cosine (as well as tangent and the co-functions) is developed using transformational geometry. This precision leads to a discussion of a mathematically *natural* unit of measurement for angle measures, a radian, and students begin to build fluency with values of sine, cosine, and tangent at $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π, and so on. The topic concludes with students graphing the sine and cosine functions and noticing various aspects of the graph, which they write down as simple trigonometric identities.

In Topic B, students make sense of periodic phenomena as they model them with trigonometric functions. They identify the periodicity, midline, and amplitude from graphs of data and use them to construct sinusoidal functions that model situations from both the biological and physical sciences. They extend the concept of polynomial identities to trigonometric identities and prove simple trigonometric identities such as the Pythagorean identity; these identities are then used to solve problems.

The module comprises 17 lessons; 3 days are reserved for administering the Mid- and End-of-Module Assessments, returning the assessments, and remediating or providing further applications of the concepts. The Mid-Module Assessment follows Topic A. The End-of-Module Assessment follows Topic B.

FOCUS STANDARDS

Analyze functions using different representations.

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Extend the domain of trigonometric functions using the unit circle.

F-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Model periodic phenomena with trigonometric functions.

F-TF.B.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

Prove and apply trigonometric identities.

F-TF.C.8 Prove the Pythagorean identity $\sin^2(\theta)+\cos^2(\theta)=1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Summarize, represent, and interpret data on two categorical and quantitative variables.

S-ID.B.6 Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.

a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.

EXTENSION STANDARDS

Extend the domain of trigonometric functions using the unit circle.

F-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$ and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

Prove and apply trigonometric identities.

F-TF.C.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

FOUNDATIONAL STANDARDS

Reason quantitatively and use units to solve problems.

N-Q.A.2 Define appropriate quantities for the purpose of descriptive modeling.

Understand the concept of a function and use function notation.

F-IF.A.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

F-IF.A.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Build a function that models a relationship between two quantities.

F-BF.A.1 Write a function that describes a relationship between two quantities.★

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*

c. (+) Compose functions. *For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.*

Build new functions from existing functions.

F-BF.B.3 Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Define trigonometric ratios and solve problems involving right triangles.

G-SRT.C.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

G-SRT.C.7 Explain and use the relationship between the sine and cosine of complementary angles.

G-SRT.C.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

FOCUS STANDARDS FOR MATHEMATICAL PRACTICE

MP.1 *Make sense of problems and persevere in solving them.* Students look for entry points into studying the “height” of the sun above the ground, first by realizing that no such quantity exists and then by surmising that the notion can still be profitably analyzed in terms of trigonometric ratios. They use this and other concrete situations to extend concepts of trigonometry studied in previous years, which were initially limited to angles between 0 and 90 degrees, to the full range of inputs; they also solve challenges about circular motion.

MP.2 *Reason abstractly and quantitatively.* Students extend the study of trigonometry to the domain of all (or almost all) real inputs. By focusing only on the linear components of circular motion (the vertical or horizontal displacement of a point in orbit), students develop the means to analyze periodic phenomena. Students also extend a classic proof of the Pythagorean theorem to discover trigonometric addition formulas.

MP.3 *Construct viable arguments and critique the reasoning of others.* The vertical and horizontal displacements of a Ferris wheel passenger car are both periodic. Students conjecture how these functions are related to the trigonometric ratios they studied in Geometry, making plausible arguments by modeling the Ferris wheel with a circle in the coordinate plane. Also, students construct valid arguments to extend trigonometric identities to the full range of inputs.

MP.4 *Model with mathematics.* The main modeling activity of this module is to analyze the vertical and horizontal displacement of a passenger car of a Ferris wheel. As students make assumptions and simplify the situation, they discover the need for sine and cosine functions

to model the periodic motion using sinusoidal functions. Students then model a large number of other periodic phenomena by fitting sinusoidal functions to data given about tides, sound waves, and daylight hours; they then solve problems using those functions in the context of those data.

MP.7 *Look for and make use of structure.* Students recognize the periodic nature of a phenomenon and look for suitable values of midline, amplitude, and frequency for it. The periodicity and properties of cyclical motion shown in graphs helps students recognize different trigonometric identities, and structure in standard proofs (of the Pythagorean theorem, for example) provides the means to extend familiar trigonometric results to a wider range of input values.

MP.8 *Look for and express regularity in repeated reasoning.* In repeatedly graphing different sinusoidal functions, students identify how parameters within the function give information about the amplitude, midline, and frequency of the function. They express this regularity in terms of a general formula for sinusoidal functions and use the formula to quickly write functions that model periodic data.

MODULE TOPIC SUMMARIES

Topic A: The Story of Trigonometry and Its Contexts

In Topic A, students develop an understanding of the six basic trigonometric functions as functions of the amount of rotation of a point on the unit circle and then translate that understanding to the trigonometric functions as functions on the real number line. In Lessons 1 and 2, a Ferris wheel provides a familiar context for the introduction of periodic functions that lead to the sine and cosine functions in Lessons 4 and 5. Lesson 1 is an exploratory lesson in which students model the circular motion of a Ferris wheel using a paper plate. The goal is to study the vertical component of the circular motion with respect to the degrees of rotation of the wheel from the initial position. This function is temporarily described as the *height function* of a passenger car on the Ferris wheel, and students produce a graph of the height function from their model. In this first lesson, students begin to understand the periodicity of the height function as the Ferris wheel completes multiple rotations (MP.7).

Lesson 2 introduces the *co-height function*, which describes the horizontal component of the circular motion of the Ferris wheel. Students again model the position of a car on a rotating Ferris wheel using a paper plate, this time with emphasis on the horizontal motion of the car. In the first lesson, heights were measured from the "ground" to the passenger car of the Ferris wheel so that the graph of the height function was contained within the first quadrant of the Cartesian plane. In this second lesson, we change our frame of reference so that the values of the height and co-height functions oscillate between $-r$ and r, where r is the radius of the wheel, inching the height and co-height functions toward the sine and cosine functions. The goal of these first two lessons is to provide a familiar context for circular motion so that students can begin to see how the horizontal and vertical components of the position of a point rotating around a circle can be described by periodic functions of the amount of rotation. Reference is made to this context as needed throughout the module.

Lesson 3 provides historical background on the development of the sine and cosine functions in India around 500 C.E. In this lesson, students generate part of a sine table and use it to calculate the positions of the sun in the sky, assuming the historical model of the sun

following a circular orbit around Earth. This lesson provides a second example of circular motion that can be modeled using the sine and cosine functions. In this lesson, the link is made between the assumed circular motion of stars and the sun and the periodic sine and cosine functions, and that link is formalized in Lesson 4.

Lesson 4 draws connections between the height function of a Ferris wheel and the sine and cosine functions used in triangle trigonometry in Geometry. This lesson extends the domain of the sine and cosine functions from the restricted domain $(0, 90)$ of degree measures of acute angles in triangles to the interval $(0, 360)$. Abstracting the sine and cosine from the height and co-height functions of the Ferris wheel allows students to practice MP.2.

In fully developing F-TF.A.2 on extending the trigonometric functions to the entire real line in Lesson 5, students need to come to know enough values of these functions to generate graphs of these functions and discern structure and properties about them (in much the same way that students were first introduced to exponential functions by studying their values at integer inputs). The most important values to learn, of course, are the values of sine and cosine functions of the most commonly used reference points: the sine and cosine of degree measures that are multiples of 30 and 45. This knowledge, in turn, serves as a concrete example for learning standard F-TF.A.1.

Lessons 6 and 7 introduce the tangent and secant functions through their geometric descriptions on a circle and link those geometric descriptions to the appropriate ratios of sine and cosine. The remaining trigonometric functions, cotangent and cosecant, are also introduced.

In Lesson 8, students construct a graph of the sine and cosine functions as functions on the real line by measuring the horizontal and vertical components of a point on the unit circle, breaking a piece of spaghetti to the appropriate length, and gluing it to the graph. Physically creating the graphs using direct measurement ties together the definition of $\sin(\theta°)$ as the y-coordinate of the point on the unit circle that has been rotated θ degrees about the origin from the point $(1, 0)$ and the value of the periodic function $f(\theta) = \sin(\theta°)$.

Lesson 9 introduces radian measure. We justify the switch to radians by drawing the graph of $y = \sin(x°)$ with the same scale on the horizontal and vertical axes, which is nearly impossible to draw. This somewhat artificial task serves many different purposes: it provides justification for the use of radian measures without referring directly to ideas of calculus, it foreshadows the lessons to come in Topic B on transforming the graph of the sine function, and it allows students to look for patterns. Students practice MP.7 when they discover the effects of changing the parameters on the graph, and they practice MP.8 when they repeatedly draw graphs of sinusoidal functions to notice the patterns. Drawing on their experience with graphing parabolas given by $y = kx^2$, students experiment with graphing calculators to produce graphs of $y = \sin(kx°)$ until they find that when $k \approx 57$ (or, equivalently, $k = \frac{180}{\pi}$), the line $y = x$ is tangent to $y = \sin(kx°)$ at the origin. Although we define the sine and cosine functions explicitly as functions of the amount of rotation of the initial ray composed of the nonnegative part of the x-axis, at the end of Lesson 9 students see that the measure of an angle θ in radians is the length of the arc subtended by the angle as specified by F-TF.A.1. Radian measure is used exclusively through the remaining lessons in the module.

The problem set for Lesson 9 focuses on finding the values of the sine and cosine functions for multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$, which aligns with F-TF.A.3. As students transition to this new way of measuring rotation, these reference points and their trigonometric values help students make sense of radian measure. The goal of this work, which began in the Geometry

course, is for students to fluently and automatically recall (or be able to derive) these values in the Precalculus and Advanced Topics course, thereby satisfying the expectation of F-TF.A.3.

The topic culminates with Lesson 10, which incorporates such identities as $\sin(\pi - x) = \sin(x)$ and $\cos(2\pi - x) = \cos(x)$ for all real numbers x into an introduction to trigonometric identities that will be studied further in Topic B. In this lesson, students analyze the graphs of the sine and cosine function and note some basic properties that are apparent from the graphs and from the unit circle, such as the periodicity of sine and cosine, the even and odd properties of the functions, and the fact that the graph of the cosine function is a horizontal shift of the graph of the sine function. Students also note the intercepts and end behavior of these graphs.

Focus Standards:	F-IF.C.7e	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
	F-TF.A.1	Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
	F-TF.A.2	Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
Instructional Days:	10	

Student Outcomes

Lesson 1: Ferris Wheels—Tracking the Height of a Passenger Car

- Students apply geometric concepts in modeling situations. Specifically, they find distances between points of a circle and a given line to represent the height above the ground of a passenger car on a Ferris wheel as it is rotated a number of degrees about the origin from an initial reference point.
- Students sketch the graph of a nonlinear relationship between variables.

Lesson 2: The Height and Co-Height Functions of a Ferris Wheel

- Students model and graph two functions given by the location of a passenger car on a Ferris wheel as it is rotated a number of degrees about the origin from an initial reference position.

Lesson 3: The Motion of the Moon, Sun, and Stars—Motivating Mathematics

- Students explore the historical context of trigonometry as a motion of celestial bodies in a presumed circular arc.
- Students describe the position of an object along a line of sight in the context of circular motion.
- Students understand the naming of the quadrants and why counterclockwise motion is deemed the positive direction of rotation in mathematics.

Lesson 4: From Circle-ometry to Trigonometry

- Students define sine and cosine as functions for degrees of rotation of the ray formed by the positive x-axis up to one full turn.
- Students use special triangles to geometrically determine the values of sine and cosine for 30, 45, 60, and 90 degrees.

Lesson 5: Extending the Domain of Sine and Cosine to All Real Numbers

- Students define sine and cosine as functions for all real numbers measured in degrees.
- Students evaluate the sine and cosine functions at multiples of 30 and 45.

Lesson 6: Why Call It Tangent?

- Students define the tangent function and understand the historical reason for its name.
- Students use special triangles to determine geometrically the values of the tangent function for 30, 45, and 60 degrees.

Lesson 7: Secant and the Co-Functions

- Students define the secant function and the co-functions in terms of points on the unit circle. They relate the names for these functions to the geometric relationships among lines, angles, and right triangles in a unit circle diagram.
- Students use reciprocal relationships to relate the trigonometric functions to each other and use these relationships to evaluate trigonometric functions at multiples of 30, 45, and 60 degrees.

Lesson 8: Graphing the Sine and Cosine Functions

- Students graph the sine and cosine functions and analyze the shape of these curves.
- For the sine and cosine functions, students sketch graphs showing key features, which include intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maxima and minima; symmetries; end behavior; and periodicity.

Lesson 9: Awkward! Who Chose the Number 360, Anyway?

- Students explore horizontal scalings of the graph of $y = \sin(x°)$.
- Students convert between degrees and radians.

Lesson 10: Basic Trigonometric Identities from Graphs

- Students observe identities from graphs of sine and cosine basic trigonometric identities and relate those identities to periodicity, even and odd properties, intercepts, end behavior, and cosine as a horizontal translation of sine.

Topic B: Understanding Trigonometric Functions and Putting Them to Use

In Topic A, students developed the ideas behind the six basic trigonometric functions, focusing primarily on the sine function. In Topic B, students use trigonometric functions to model periodic behavior. We end the module with the study of trigonometric identities and how to prove them.

Lesson 11 continues the idea started in Lesson 9 in which students graphed $y = \sin(kx°)$ for different values of k. In Lesson 11, teams of students work to understand the effect of changing the parameters A, ω, h, and k in the graph of the function $y = \mathrm{A}(\sin(\omega(x-h))) + k$, so that in Lesson 12 students can fit sinusoidal functions to given scenarios, which aligns with F-IF.C.7e and F-TF.B.5. While Lesson 12 requires that students find a formula that precisely models periodic motion in a given scenario, Lesson 13 is distinguished by nonexact modeling, as in S-ID.B.6a. In Lesson 13, students analyze given real-world data and fit the data with an appropriate sinusoidal function, providing authentic practice with MP.3 and MP.4 as they debate about appropriate choices of functions and parameters.

Lesson 14 returns to the idea of graphing functions on the real line and producing graphs of $y = \tan(x)$. Students work in groups to produce the graph of one branch of the tangent function by plotting points on a specified interval. The individual graphs are compiled into one classroom graph to emphasize the periodicity and basic properties of the tangent function.

To wrap up the module, students revisit the idea of mathematical proof in Lessons 15–17. Lesson 15 aligns with standard F-TF.C.8, proving the Pythagorean identity. In Lesson 17, students discover the formula for $\sin(\alpha + \beta)$ using MP.8, in alignment with standard F-TF.B.9(+), but teachers may choose to present the optional rigorous proof of this formula that is provided in the lesson. Standard F-TF.B.9(+) is included because it logically coheres with the rest of the content in the module. Throughout Lessons 15, 16, and 17, the emphasis is on the proper statement of a trigonometric identity as the pairing of a statement that two functions are equivalent on a given domain and an identification of that domain. For example, the identity "$\sin^2(\theta) + \cos^2(\theta) = 1$ for all real numbers θ" is a statement that the two functions $f_1(\theta) = \sin^2(\theta) + \cos^2(\theta)$ and $f_2(\theta) = 1$ have the same value for every real number θ. As students revisit the idea of proof in these lessons, they are prompted to follow the steps of writing a valid proof:

1. Define the variables. For example, "Let θ be any real number."
2. Establish the identity by starting with the expression on one side of the equation and transforming it into the expression on the other side through a sequence of algebraic steps using rules of logic, algebra, and previously established identities. For example:

$$\begin{aligned}\cos(2\theta) &= \cos(\theta + \theta)\\ &= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta)\\ &= \cos^2(\theta) - \sin^2(\theta).\end{aligned}$$

3. Conclude the proof by stating the identity in its entirety, both the statement and the domain. For example, "Then $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ for any real number θ."

Focus Standards:	F-IF.C.7e	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.* e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
	F-TF.B.5	Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.
	F-TF.C.8	Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.
	S-ID.B.6a	Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.
Instructional Days:	7	

Student Outcomes

Lesson 11: Transforming the Graph of the Sine Function

- Students formalize the period, frequency, phase shift, midline, and amplitude of a general sinusoidal function by understanding how the parameters A, ω, h, and k in the formula

$$f(x) = A\sin(\omega(x - h)) + k$$

are used to transform the graph of the sine function and how variations in these constants change the shape and position of the graph of the sine function.

- Students learn the relationship among the constants A, ω, h, and k in the formula $f(x) = \mathrm{A}\sin(\omega(x-h))+k$ and the properties of the sine graph. In particular, they learn that
  - $|\mathrm{A}|$ is the *amplitude* of the function. The amplitude is the distance between a maximal point of the graph of the sinusoidal function and the midline (i.e., $\mathrm{A} = f_{max} - k$ or $\mathrm{A} = \frac{f_{max}-f_{min}}{2}$).
  - $\frac{2\pi}{|\omega|}$ is the *period* of the function. The period P is the distance between two consecutive maximal points (or two consecutive minimal points) on the graph of the sinusoidal function. Thus, $\omega = \frac{2\pi}{\mathrm{P}}$.
  - $\frac{\omega}{2\pi}$ is the *frequency* of the function (the frequency is the reciprocal of the period).
  - h is called the *phase shift*.
  - The graph of $y = k$ is called the *midline*.
  - Furthermore, the graph of the sinusoidal function f is obtained by vertically scaling the graph of the sine function by A, horizontally scaling the resulting graph by $\frac{1}{\omega}$, and then horizontally and vertically translating the resulting graph by h and k units, respectively.

Lesson 12: Ferris Wheels—Using Trigonometric Functions to Model Cyclical Behavior

- Students review how changing the parameters A, ω, h, and k in

$$f(x) = \mathrm{A}\sin(\omega(x-h))+k$$

  affects the graph of a sinusoidal function.
- Students examine the example of the Ferris wheel, using height, distance from the ground, period, and so on, to write a function of the height of the passenger cars in terms of the sine function:

$$f(x) = \mathrm{A}\sin(\omega(x-h))+k$$

Lesson 13: Tides, Sound Waves, and Stock Markets

- Students model cyclical phenomena from biological and physical science using trigonometric functions.
- Students understand that some periodic behavior is too complicated to be modeled by simple trigonometric functions.

Lesson 14: Graphing the Tangent Function

- Students graph the tangent function.
- Students use the unit circle to express the values of the tangent function for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of $\tan(x)$, where x is any real number in the domain of the tangent function.

Lesson 15: What Is a Trigonometric Identity?

- Students prove the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$.
- Students extend trigonometric identities to the real line, with attention to domain and range.
- Students use the Pythagorean identity to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$, given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the terminal ray of the rotation.

Lesson 16: Proving Trigonometric Identities

- Students prove simple identities involving the sine function, cosine function, and secant function.
- Students recognize features of proofs of identities.

Lesson 17: Trigonometric Identity Proofs

- Students see derivations and proofs of the addition and subtraction formulas for sine and cosine.
- Students prove some simple trigonometric identities.

MODULE 3: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

OVERVIEW

In this module, students synthesize and generalize what they have learned about a variety of function families. They extend the domain of exponential functions to the entire real line (N-RN.A.1) and then extend their work with these functions to include solving exponential equations with logarithms (F-LE.A.4). They use appropriate tools to explore the effects of transformations on graphs of exponential and logarithmic functions. They notice that the transformations of a graph of a logarithmic function relate to the logarithmic properties (F-BF.B.3). Students identify appropriate types of functions to model a situation. They adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as "the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (see p. 72 of CCSSM) is at the heart of this module. In particular, through repeated opportunities working through the modeling cycle, students acquire the insight that the same mathematical or statistical structure can sometimes model seemingly different situations.

This module builds on the work in Algebra I Modules 3 and 5, where students first modeled situations using exponential functions and considered which type of function would best model a given real-world situation. The module also introduces students to the extension standards relating to inverse functions and composition of functions to further enhance student understanding of logarithms.

Topic E is a culminating project spread out over several lessons in which students consider applying their knowledge to financial literacy. They plan a budget, consider borrowing money to buy a car and a home, study paying off a credit card balance, and decide how they could accumulate one million dollars.

The module comprises 33 lessons; 12 days are reserved for administering the Mid- and End-of-Module Assessments, returning the assessments, and remediating or providing further applications of the concepts. The Mid-Module Assessment follows Topic B. The End-of-Module Assessment follows Topic E.

FOCUS STANDARDS

Extend the properties of exponents to rational exponents.

N-RN.A.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{\frac{1}{3}}$ *to be the cube root of* 5 *because we want* $(5^{\frac{1}{3}})^3 = 5^{(\frac{1}{3})^3}$ *to hold, so* $(5^{\frac{1}{3}})^3$ *must equal* 5.

N-RN.A.2[9] Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Reason quantitatively and use units to solve problems.

N-Q.A.2[10] Define appropriate quantities for the purpose of descriptive modeling.★

Write expressions in equivalent forms to solve problems.

A-SSE.B.3[11] Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is* 15%.

A-SSE.B.4[12] Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★

Create equations that describe numbers or relationships.

A-CED.A.1[13] Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

Represent and solve equations and inequalities graphically.

A-REI.D.11[14] Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

Understand the concept of a function and use function notation.

F-IF.A.3[15] Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $\mathrm{f}(0) = \mathrm{f}(1) = 1$, $\mathrm{f}(\mathrm{n}+1) = \mathrm{f}(\mathrm{n}) + \mathrm{f}(\mathrm{n}-1)$ *for* $\mathrm{n} \geq 1$.

Interpret functions that arise in applications in terms of the context.

F-IF.B.4[16] For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F-IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function* h(n) *gives the number of person-hours it takes to assemble* n *engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F-IF.B.6[17] Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Analyze functions using different representations.

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F-IF.C.8[18] Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*

F-IF.C.9[19] Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Build a function that models a relationship between two quantities.

F-BF.A.1 Write a function that describes a relationship between two quantities.★

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.[20]

b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*[21]

F-BF.A.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

Build new functions from existing functions.

F-BF.B.3[22] Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

F-BF.B.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x+1)/(x-1)$ *for* $x \neq 1$.

Construct and compare linear, quadratic, and exponential models and solve problems.

F-LE.A.2[23] Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★

F-LE.A.4[24] For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

Interpret expressions for functions in terms of the situation they model.

F-LE.B.5[25] Interpret the parameters in a linear or exponential function in terms of a context.★

FOUNDATIONAL STANDARDS

Use properties of rational and irrational numbers.

N-RN.B.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Interpret the structure of expressions.

A-SSE.A.2 Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

Create equations that describe numbers or relationships.

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance* R.★

Represent and solve equations and inequalities graphically.

A-REI.D.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Understand the concept of a function and use function notation.

F-IF.A.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

F-IF.A.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Construct and compare linear, quadratic, and exponential models and solve problems.

F-LE.A.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.★

b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

F-LE.A.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

FOCUS STANDARDS FOR MATHEMATICAL PRACTICE

MP.1 *Make sense of problems and persevere in solving them.* Students make sense of rational and real number exponents and in doing so are able to apply exponential functions to solve problems involving exponential growth and decay for continuous domains such as time.

They explore logarithms numerically and graphically to understand their meaning and how they can be used to solve exponential equations. Students have multiple opportunities to make connections between information presented graphically, numerically, and algebraically and search for similarities between these representations to further understand the underlying mathematical properties of exponents and logarithms. When presented with a wide variety of information related to financial planning, students make sense of the given information and use appropriate formulas to effectively plan for a long-term budget and savings plan.

MP.2 *Reason abstractly and quantitatively.* Students consider appropriate units when exploring the properties of exponents for very large and very small numbers. They reason about quantities when solving a wide variety of problems that can be modeled using logarithms or exponential functions. Students relate the parameters in exponential expressions to the situations they model. Students write and solve equations and then interpret their solutions within the context of a problem.

MP.4 *Model with mathematics.* Students use exponential functions to model situations involving exponential growth and decay. They model the number of digits needed to assign identifiers using logarithms. They model exponential growth using a simulation with collected data. The application of exponential functions and logarithms as a means to solve an exponential equation is a focus of several lessons that deal with financial literacy and planning a budget. Here, students must make sense of several different quantities and their relationships as they plan and prioritize for their future financial solvency.

MP.7 *Look for and make use of structure.* Students extend the laws of exponents for integer exponents to rational and real number exponents. They connect how these laws are related to the properties of logarithms and understand how to rearrange an exponential equation into logarithmic form. Students analyze the structure of exponential and logarithmic functions to understand how to sketch graphs and see how the properties relate to transformations of these types of functions. They analyze the structure of expressions to reveal properties, such as recognizing when a function models exponential growth versus decay. Students use the structure of equations to understand how to identify an appropriate solution method.

MP.8 *Look for and express regularity in repeated reasoning.* Students discover the properties of logarithms and the meaning of a logarithm by investigating numeric examples. They develop formulas that involve exponentials and logarithms by extending patterns and examining tables and graphs. Students generalize transformations of graphs of logarithmic functions by examining several different cases.

MODULE TOPIC SUMMARIES

Topic A: Real Numbers

In Topic A, students prepare to generalize what they know about various function families by examining the behavior of exponential functions. One goal of the module is to show that the domain of the exponential function $f(x) = b^x$, where b is a positive number not equal to 1, is all real numbers. In Lesson 1, students review and practice applying the laws of exponents to expressions in which the exponents are integers. Students first tackle a challenge problem on paper folding that is related to exponential growth and then apply and practice applying the laws of exponents to rewriting algebraic expressions. They experiment, create a table of values, observe patterns, and then generalize a formula to represent different measurements in the folded stack of paper as specified in F-LE.A.2. They also use the laws of exponents to work with very large and very small numbers.

Lesson 2 sets the stage for the introduction of base-10 logarithms in Topic B of the module by reviewing how to express numbers using scientific notation, how to compute using scientific notation, and how to use the laws of exponents to simplify those computations, in accordance with N-RN.A.2. Students should gain a sense of the change in magnitude when different powers of 10 are compared. The activities in these lessons prepare students for working with quantities that increase in magnitude by powers of 10 and show them the usefulness of exponent properties when performing arithmetic operations. Similar work is done in later lessons relating to logarithms. Exercises on distances between planets in the solar system and on comparing magnitudes in other real-world contexts provide additional practice with arithmetic operations on numbers written using scientific notation.

Lesson 3 begins with students examining the graph of $y = 2^x$ and estimating values as a means of extending their understanding of integer exponents to rational exponents. The examples are generalized to $2^{\frac{1}{n}}$ before generalizing further to $2^{\frac{m}{n}}$. As the domain of the identities involving exponents is expanded, it is important to maintain consistency with the properties already developed. Students work specifically to make sense that $2^{\frac{1}{2}} = \sqrt{2}$ and $2^{\frac{1}{3}} = \sqrt[3]{2}$ to develop the more general concept that $2^{\frac{1}{n}} = \sqrt[n]{2}$. The lesson demonstrates how people develop mathematics (1) to be consistent with what is already known and (2) to make additional progress. In addition, students practice MP.7 as they extend the rules for integer exponents to rules for rational exponents (N-RN.A.1).

Lesson 4 continues the discussion of properties of exponents and radicals, and students continue to practice MP.7 as they extend their understanding of exponents to all rational numbers and for all positive real bases as specified in N-RN.A.1. Students rewrite expressions involving radicals and rational exponents using the properties of exponents (N-RN.A.2). The notation $x^{\frac{1}{n}}$ specifically indicates the principal root of x: the positive root when n is even and the real-valued root when n is odd. To avoid inconsistencies in the later work with logarithms, x is required to be positive.

Lesson 5 revisits the work of Lesson 3 and extends student understanding of the domain of the exponential function $f(x) = b^x$, where b is a positive real number, from the rational numbers to all real numbers through the process of considering what it means to raise a number to an irrational exponent (such as $2^{\sqrt{2}}$). In many ways, this lesson parallels the work students did in Lesson 3 to make a solid case for why the laws of exponents hold for all rational number exponents. The recursive procedure that students employ in this lesson aligns with F-BF.A.1a. This lesson is important both because it helps portray mathematics as a coherent body of knowledge that makes sense and because it is necessary to make sure that students understand that logarithms can be irrational numbers. Essentially, it is necessary to guarantee that exponential and logarithmic functions are continuous functions. Students take away from these lessons an understanding that the domain of exponents in the laws of exponents does indeed extend to all real numbers rather than just to the integers, as defined previously in Grade 8.

Lesson 6 is a modeling lesson in which students practice MP.4 when they find an exponential function to model the amount of water in a tank after t seconds when the height of the water is constantly doubling or tripling and in which students apply F-IF.B.6 as they explore the average rate of change of the height of the water over smaller and smaller intervals. If the height of the water in the tank at time t seconds is denoted by $H(t) = b^t$, then the average rate of change of the height of the water on an interval $[T, T + \varepsilon]$ is approximated by $\frac{H(T+\varepsilon)-H(t)}{\varepsilon} \approx c \cdot H(T)$. Students calculate that if the height of the water is doubling each second, then $c \approx 0.69$, and if the height of the water is tripling each second, then $c \approx 1.1$.

Students discover Euler's number, e, by applying repeated reasoning (MP.8) and numerically approximating the base b for which the constant c is equal to 1. Euler's number is used extensively in the future and occurs in many different applications.

Focus Standards:	N-RN.A.1	Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{1/3}$ *to be the cube root of* 5 *because we want* $(5^{1/3})^3 = 5^{(1/3)3}$ *to hold, so* $(5^{1/3})^3$ *must equal* 5.
	N-RN.A.2	Rewrite expressions involving radicals and rational exponents using the properties of exponents.
	N-Q.A.2	Define appropriate quantities for the purpose of descriptive modeling.★
	F-IF.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★
	F-BF.A.1a	Write a function that describes a relationship between two quantities.★ a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
	F-LE.A.2	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★
Instructional Days:	6	

Student Outcomes

Lesson 1: Integer Exponents

- Students review and practice applying the properties of exponents for integer exponents.
- Students model a real-world scenario involving exponential growth and decay.

Lesson 2: Base 10 and Scientific Notation

- Students review place value and scientific notation.
- Students use scientific notation to compute with large numbers.

Lesson 3: Rational Exponents—What are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$?

- Students calculate quantities that involve positive and negative rational exponents.

Lesson 4: Properties of Exponents and Radicals

- Students rewrite expressions involving radicals and rational exponents using the properties of exponents.

Lesson 5: Irrational Exponents—What are $2^{\sqrt{2}}$ and 2^{π}?

- Students approximate the value of quantities that involve positive irrational exponents.
- Students extend the domain of the function $f(x) = b^x$ for positive real numbers b to all real numbers.

Lesson 6: Euler's Number, e

- Students write an exponential function that represents the amount of water in a tank after t seconds if the height of the water doubles every 10 seconds.
- Students discover Euler's number e by numerically approaching the constant for which the height of water in a tank equals the rate of change of the height of the water in the tank.
- Students calculate the average rate of change of a function.

Topic B: Logarithms

The lessons covered in Topic A familiarize students with the laws and properties of real-valued exponents. In Topic B, students extend their work with exponential functions to include solving exponential equations numerically and developing an understanding of the relationship between logarithms and exponentials. In Lesson 7, students use an algorithmic numerical approach to solve simple exponential equations that arise from modeling the growth of bacteria and other populations (F-BF.A.1a). Students work to develop progressively better approximations for the solutions to equations whose solutions are irrational numbers. In doing this, students increase their understanding of the real number system and truly begin to understand what it means for a number to be irrational. Students learn that some simple exponential equations can be solved exactly without much difficulty, but that mathematical tools are lacking to solve other equations whose solutions must be approximated numerically.

Lesson 8 begins with the logarithmic function disguised as the more intuitive "WhatPower" function, whose behavior is studied as a means of introducing how the function works and what it does to expressions. Students find the power needed to raise a base b in order to produce a given number. The lesson ends with students defining the term *logarithm base* b. Lesson 8 is just a first introduction to logarithms in preparation for solving exponential equations per F-LE.A.4; students neither use tables nor look at graphs in this lesson. Instead, they simply develop the ideas and notation of logarithmic expressions, leaving many ideas to be explored later in the module.

Just as population growth is a natural example that gives context to exponential growth, Lesson 9 gives context to logarithmic calculation through the example of assigning unique identification numbers to a group of people. In this lesson, students consider the meaning of the logarithm in the context of calculating the number of digits needed to create student ID numbers, phone numbers, and Social Security numbers, in accordance with N-Q.A.2. This gives students a real-world context for the abstract idea of a logarithm; in particular, students observe that a base-10 logarithm provides a way to keep track of the number of digits used in a number in the base-10 system.

Lessons 10–15 develop both the theory of logarithms and procedures for solving various forms of exponential and logarithmic equations. In Lessons 10 and 11, students discover the logarithmic properties by completing carefully structured logarithmic tables and answering sets of directed questions. Throughout these two lessons, students look for structure in the table and use that structure to extract logarithmic properties (MP.7). Using the structure of the logarithmic expression together with the logarithmic properties to rewrite an expression aligns with the foundational standard A-SSE.A.2. While the logarithmic properties are not themselves explicitly listed in the standards, standard F-LE.A.4 cannot be adequately met without an understanding of how to apply logarithms to solve exponential equations, and the seemingly odd behavior of graphs of logarithmic functions (F-IF.C.7e) cannot be adequately explained without an understanding of the properties of logarithms. In particular, in Lesson 11, students discover the "most important property of logarithms": For positive real numbers x and y, $\log(xy) = \log(x) + \log(y)$. Students also discover the pattern $\log_b(\frac{1}{x}) = -\log_b(x)$ that leads to conjectures about additional properties of logarithms.

Lesson 12 continues the consideration of properties of the logarithm function, while remaining focused solely on base-10 logarithms. Its centerpiece is the demonstration of basic properties of logarithms such as the power, product, and quotient properties, which allows

students to practice MP.3 and A-SSE.A.2, providing justification in terms of the definition of logarithm and the properties already developed. In this lesson, students begin to learn how to solve exponential equations, beginning with base-10 exponential equations that can be solved by taking the common logarithm of both sides of the equation.

Lesson 13 again focuses on the structure of expressions (A-SSE.A.2), as students change logarithms from one base to another. It begins by showing students how they can make that change and then develops properties of logarithms for the general base b. Students are introduced to the use of a calculator instead of a table in finding logarithms, and then *natural logarithms* are defined: $\ln(x) = \log_e(x)$. One goal of the lesson, in addition to introducing the base e for logarithms, is to explain why, for finding logarithms to any base, the calculator has only LOG and LN keys. In this lesson, students learn to solve exponential equations with any base by the application of an appropriate logarithm. Lessons 12 and 13 both address F-LE.A.4, solving equations of the form $ab^{ct} = d$, as do later lessons in the module.

Lesson 14 includes the first introduction to solving logarithmic equations. In this lesson, students apply the definition of the logarithm to rewrite logarithmic equations in exponential form, so the equations must first be rewritten in the form $\log_b(\mathrm{X}) = c$ for an algebraic expression X and some constant c. Solving equations in this way requires that students think deeply about the definition of the logarithm and how logarithms interact with exponential expressions. Although solving logarithmic equations is not listed explicitly in the standards, this skill is implicit in standard A-REI.D.11, which has students solve equations of the form $f(x) = g(x)$ where f and g can be logarithmic functions. In addition, logarithmic equations provide a greater context in which to study both the properties of logarithms and the definition, both of which are needed to solve the equations listed in F-LE.A.4.

Topic B concludes with Lesson 15, in which students learn a bit of the history of how and why logarithms first appeared. The materials for this lesson contain a base-10 logarithm table. Although modern technology has made logarithm tables functionally obsolete, there is still value in understanding the historical development of logarithms. Logarithms were critical to the development of astronomy and navigation in the days before computing machines, and this lesson presents a rationale for the pretechnological advantage afforded to scholars by the use of logarithms. In this lesson, the case is finally made that logarithm functions are one to one (without explicitly using that terminology): If $\log_b(\mathrm{X}) = \log_b(\mathrm{Y})$, then $\mathrm{X} = \mathrm{Y}$. In alignment with A-SSE.A.2, this fact not only validates the use of tables to look up anti-logarithms but also allows exponential equations to be solved with logarithms on both sides of the equation.

Focus Standards:	N-Q.A.2	Define appropriate quantities for the purpose of descriptive modeling.★
	A-CED.A.1	Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★
	F-BF.A.1a	Write a function that describes a relationship between two quantities.★ a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
	F-LE.A.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★
Instructional Days:	9	

Student Outcomes

Lesson 7: Bacteria and Exponential Growth

- Students solve simple exponential equations numerically.

Lesson 8: The “WhatPower” Function

- Students calculate a simple logarithm using the definition.

Lesson 9: Logarithms—How Many Digits Do You Need?

- Students use logarithms to determine how many characters are needed to generate unique identification numbers in different scenarios.
- Students understand that logarithms are useful when relating the number of digits in a number to the magnitude of the number and that base-10 logarithms are useful when measuring quantities that have a wide range of values, such as the magnitude of earthquakes, volume of sound, and pH levels in chemistry.

Lesson 10: Building Logarithmic Tables

- Students construct a table of logarithms base 10 and observe patterns that indicate properties of logarithms.

Lesson 11: The Most Important Property of Logarithms

- Students construct a table of logarithms base 10 and observe patterns that indicate properties of logarithms.

Lesson 12: Properties of Logarithms

- Students justify properties of logarithms using the definition and properties already developed.

Lesson 13: Changing the Base

- Students understand how to change logarithms from one base to another.
- Students calculate logarithms with any base using a calculator that computes only logarithms base 10 and base e.
- Students justify properties of logarithms with any base.

Lesson 14: Solving Logarithmic Equations

- Students solve simple logarithmic equations using the definition of logarithm and logarithmic properties.

Lesson 15: Why Were Logarithms Developed?

- Students use logarithm tables to calculate products and quotients of multi-digit numbers without technology.
- Students understand that logarithms were developed to speed up arithmetic calculations by reducing multiplication and division to the simpler operations of addition and subtraction.
- Students solve logarithmic equations of the form $\log(X) = \log(Y)$ by equating X and Y.

Topic C: Exponential and Logarithmic Functions and Their Graphs

The lessons covered in Topics A and B build on students’ prior knowledge of the properties of exponents, exponential expressions, and solving equations by extending the properties of exponents to all real number exponents and positive real number bases before introducing logarithms. This topic reintroduces exponential functions, introduces logarithmic functions, explains their inverse relationship, and explores the features of their graphs and how they can be used to model data.

Lesson 16 ties back to work in Topic A by helping students further extend their understanding of the properties of real numbers, both rational and irrational (N-RN.B.3). This Algebra I standard is revisited in Algebra II so that students know and understand that the exponential functions are defined for all real numbers, and that the graphs of the exponential functions can thus be represented by a smooth curve. Another consequence is that the logarithm functions are also defined for all positive real numbers. Lessons 17 and 18 introduce the graphs of logarithmic functions and exponential functions. Students compare the properties of graphs of logarithm functions for different bases and identify common features, which align with standards F-IF.B.4, F-IF.B.5, and F-IF.C.7. Students understand that because the range of this function is all real numbers, then some logarithms must be irrational. Students notice that the graphs of $f(x)=b^x$ and $g(x)=\log_b(x)$ appear to be related via a reflection across the graph of the equation $y=x$.

Lesson 19 addresses standards F-BF.B.4a and F-LE.A.4 while continuing the ideas introduced graphically in Lesson 18 to help students make the connection that the logarithmic function base b and the exponential function base b are inverses of each other. Inverses are introduced first by discussing operations and functions that can "undo" each other; then students look at the graphs of pairs of these functions. The lesson ties the ideas back to reflections in the plane from Geometry and illuminates why the graphs of inverse functions are reflections of each other across the line given by $y=x$, developing these ideas intuitively without formalizing what it means for two functions to be inverses. Inverse functions will be addressed in greater detail in Precalculus.

During all of these lessons, connections are made to the properties of logarithms and exponents. The relationship between graphs of these functions, the process of sketching a graph by transforming a parent function, and the properties associated with these functions are linked in Lessons 20 and 21, showcasing standards F-IF.C.7e and F-BF.B.3. Students use properties and their knowledge of transformations to explain why two seemingly different functions such as $f(x)=\log(10x)$ and $g(x)=1+\log(x)$ have the same graph. Lesson 21 revisits the natural logarithm function, and students see how the change of base property of logarithms implies that we can write a logarithm function of any base b as a vertical scaling of the natural logarithm function (or any other base logarithm function we choose).

Finally, in Lesson 22, students must synthesize knowledge across both Algebra I and Algebra II to decide whether a linear, quadratic, sinusoidal, or exponential function will best model a real-world scenario by analyzing the way in which we expect the quantity in question to change. For example, students need to determine whether or not to model daylight hours in Oslo, Norway, with a linear or a sinusoidal function because the data appear to be linear, but, in context, the choice is clear. Students model the outbreak of a flu epidemic with an exponential function and a falling body with a quadratic function. In this lesson, the majority of the scenarios that require modeling are described verbally, and students determine an explicit expression for many of the functions, in accordance with F-BF.A.1a, F-LE.A.1, and F-LE.A.2.

Focus Standards:	F-IF.B.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity*.★
	F-IF.B.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship is describes. *For example, if the function* h(n) *gives the number of person-hours it takes to assemble* n *engines in a factory, then the positive integers would be an appropriate domain for the function.*★

	F-IF.C.7e	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.* e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
	F-BF.A.1a	Write a function that describes a relationship between two quantities.* a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
	F-BF.B.3	Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.
	F-BF.B.4a	Find inverse functions. a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x + 1)/(x - 1)$ *for* $x \neq 1$.
	F-LE.A.2	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).
	F-LE.A.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers, and the base b is 2, 10, or e; evaluate the logarithm using technology.
Instructional Days:	7	

Student Outcomes

Lesson 16: Rational and Irrational Numbers

- Students interpret addition and multiplication of two irrational numbers in the context of logarithms and find better-and-better decimal approximations of the sum and product, respectively.
- Students work with and interpret logarithms with irrational values in preparation for graphing logarithmic functions.

Lesson 17: Graphing the Logarithm Function

- Students graph the functions $f(x) = \log(x)$, $g(x) = \log_2(x)$, and $h(x) = \ln(x)$ by hand and identify key features of the graphs of logarithmic functions.

Lesson 18: Graphs of Exponential Functions and Logarithmic Functions

- Students compare the graph of an exponential function to the graph of its corresponding logarithmic function.
- Students note the geometric relationship between the graph of an exponential function and the graph of its corresponding logarithmic function.

Lesson 19: The Inverse Relationship Between Logarithmic and Exponential Functions

- Students understand that the logarithmic function base b and the exponential function base b are inverse functions.

Lesson 20: Transformations of the Graphs of Logarithmic and Exponential Functions

- Students study transformations of the graphs of logarithmic functions and learn the standard form of generalized logarithmic and exponential functions.
- Students use the properties of logarithms and exponents to produce equivalent forms of exponential and logarithmic expressions. In particular, they notice that different types of transformations can produce the same graph due to these properties.

Lesson 21: The Graph of the Natural Logarithm Function

- Students understand that the change of base property allows us to write every logarithm function as a vertical scaling of a natural logarithm function.
- Students graph the natural logarithm function and understand its relationship to other base b logarithm functions. They apply transformations to sketch the graph of natural logarithm functions by hand.

Lesson 22: Choosing a Model

- Students analyze data and real-world situations and find a function to use as a model.
- Students study properties of linear, quadratic, sinusoidal, and exponential functions.

Topic D: Using Logarithms in Modeling Situations

This topic opens with a simulation and modeling activity in which students start with one bean, roll it out of a cup onto the table, and add more beans each time the marked side is up. The lesson unfolds by having students discover an exponential relationship by examining patterns when the data are presented numerically and graphically. Students blend what they know about probability and exponential functions to interpret the parameters a and b in the functions $f(t) = a(b^t)$ that they find to model their experimental data (F-LE.B.5, A-CED.A.2).

In both Algebra I and Lesson 6 in this module, students had to solve exponential equations when modeling real-world situations numerically or graphically. Lesson 24 shows students how to use logarithms to solve these types of equations analytically and makes explicit the connections between numeric, graphical, and analytical approaches, invoking the related standards F-LE.A.4, F-BF.B.4a, and A-REI.D.11. Students are encouraged to use multiple approaches to solve equations generated in the next several lessons.

In Lessons 25–27, a general growth/decay rate formula is presented to students to help construct models from data and descriptions of situations. Students must use properties of exponents to rewrite exponential expressions in order to interpret the properties of the function (F-IF.C.8b). For example, in Lesson 27, students compare the initial populations and annual growth rates of population functions given in the forms $E(t) = 281.4(1.0093)^{t-100}$, $f(t) = 81.1(1.0126)^t$, and $g(t) = 76.2(13.6)^{\frac{t}{10}}$. Many of the situations and problems presented here were first encountered in Module 3 of Algebra I; students are now able to solve equations involving exponents that they could only estimate previously, such as finding the time when the population of the United States is expected to surpass a half-billion people. Students answer application questions in the context of the situation and use technology to evaluate logarithms of base 10 and e. In addition, Lesson 25 begins to develop geometric sequences that are needed for the financial content in the next topic (F-BF.A.2). Lesson 26 continues developing the skills of distinguishing between situations that require exponential models and those that require linear models (F-LE.A.1), and Lesson 27 continues the work with geometric sequences that started in Lesson 25 (F-IF.B.3, F-BF.A.1a).

Lesson 28 closes this topic and addresses F-BF.A.1b by revisiting Newton's law of cooling, a formula that involves the sum of an exponential function and a constant function. Students first learned about this formula in Algebra I, but now that they are armed with logarithms and have more experience understanding how transformations affect the graph of a function, they can find the precise value of the decay constant using logarithms and thus can solve problems related to this formula more precisely and with greater depth of understanding.

Focus Standards:	A-SSE.B.3c	Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is* 15%.
	A-CED.A.1	Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*
	A-REI.D.11	Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*
	F-IF.B.3	Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ *for* $n \geq 1$.
	F-IF.B.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*
	F-IF.C.8b	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*
	F-IF.C.9	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*
	F-BF.A.1	Write a function that describes a relationship between two quantities.* a. Determine an explicit expression, a recursive process, or steps for calculation from a context. b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*
	F-BF.A.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.*
	F-BF.B.4a	Find inverse functions. a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x + 1)/(x - 1)$ *for* $x \neq 1$.
	F-LE.A.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.
	F-LE.B.5	Interpret the parameters in a linear or exponential function in terms of a context.
Instructional Days:	6	

Student Outcomes

Lesson 23: Bean Counting

- Students gather experimental data and determine which type of function is best to model the data.
- Students use properties of exponents to interpret expressions for exponential functions.

Lesson 24: Solving Exponential Equations

- Students apply properties of logarithms to solve exponential equations.
- Students relate solutions to $f(x) = g(x)$ to the intersection point(s) on the graphs of $y = f(x)$ and $y = g(x)$ in the case where f and g are constant or exponential functions.

Lesson 25: Geometric Sequences and Exponential Growth and Decay

- Students use geometric sequences to model situations of exponential growth and decay.
- Students write geometric sequences explicitly and recursively and translate between the two forms.

Lesson 26: Percent Rate of Change

- Students develop a *general growth/decay rate formula* in the context of compound interest.
- Students compute *future values* of investments with continually compounding interest rates.

Lesson 27: Modeling with Exponential Functions

- Students create exponential functions to model real-world situations.
- Students use logarithms to solve equations of the form $f(t) = a \cdot b^{ct}$ for t.
- Students decide which type of model is appropriate by analyzing numerical or graphical data and verbal descriptions and by comparing different data representations.

Lesson 28: Newton's Law of Cooling, Revisited

- Students apply knowledge of exponential and logarithmic functions and transformations of functions to a contextual situation.

Topic E: Geometric Series and Finance

Topic E is a culminating series of lessons driven by MP.4 (model with mathematics). Students apply what they have learned about mathematical models and exponential growth to financial literacy, while developing and practicing the formula for the sum of a finite geometric series. Lesson 29 develops the future value formula for a structured savings plan and, in the process, develops the formula for the sum of a finite geometric series (A-SSE.B.4). The summation symbol, Σ, is introduced in this lesson.

Lesson 30 introduces loans through the context of purchasing a car. To develop the formula for the present value of an annuity, students combine two formulas for the future value of the annuity (F-BF.A.1b) and apply the sum of a finite geometric series formula. Throughout the remaining lessons, various forms of the present value of an annuity formula are used to calculate monthly payments and loan balances. The comparison of the effects of various interest rates and repayment schedules requires that students translate between symbolic and numerical representations of functions (F-IF.C.9). Lesson 31 addresses the issue of revolving credit such as credit cards, for which the borrower can choose how much of the debt to pay each cycle. Students again sum a geometric series to develop a formula for this scenario, and it turns out to be equivalent to the formula used for car loans. Key features of tables and graphs are used to answer questions about finances (F-IF.C.7e).

Lessons 32 and 33 are modeling lessons in which students apply what they have learned in earlier lessons to new financial situations (MP.4). Lesson 32 may be extended to an open-ended project in which students research buying a home and justify its affordability. Lesson 33, the final lesson of the module, is primarily a summative lesson in which students formulate a plan to have $1 million in assets within a fixed time frame, using the formulas developed in

the prior lessons in the topic. Students graph the present value function and compare that with an amortization table, in accordance with F-IF.C.9. In both of these lessons, students need to combine functions using standard arithmetic operations (F-IF.A.1b).

Focus Standards:	A-SSE.B.4	Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*
	F-IF.C.7e	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.* e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
	F-IF.C.8b	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay*.
	F-IF.C.9	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*
	F-BF.A.1b	Write a function that describes a relationship between two quantities.* b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*
	F-BF.A.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.*
	F-LE.B.5	Interpret the parameters in a linear or exponential function in terms of a context.
Instructional Days:	5	

Student Outcomes

Lesson 29: The Mathematics Behind a Structured Savings Plan

- Students derive the sum of a finite geometric series formula.
- Students apply the sum of a finite geometric series formula to a structured savings plan.

Lesson 30: Buying a Car

- Students use the sum of a finite geometric series formula to develop a formula to calculate a payment plan for a car loan and use that calculation to derive the present value of an annuity formula.

Lesson 31: Credit Cards

- Students compare payment strategies for a decreasing credit card balance.
- Students apply the sum of a finite geometric series formula to a decreasing balance on a credit card.

Lesson 32: Buying a House

- Students model the scenario of buying a house.
- Students recognize that a mortgage is mathematically equivalent to car loans studied in Lesson 30 and apply the present value of annuity formula to a new situation.

Lesson 33: The Million Dollar Problem

- Students use geometric series to calculate how much money should be saved each month to have $1 million in assets within a specified amount of time.

MODULE 4: INFERENCES AND CONCLUSIONS FROM DATA

OVERVIEW

The concepts of probability and statistics covered in Algebra II build on students' previous work in Grade 7 and Algebra I. Topics A and B address standards S-CP.A.1–5 and S-CP.B.6–7, which deal primarily with probability. In Topic A, fundamental ideas from Grade 7 are revisited and extended to allow students to build a more formal understanding of probability. More complex events are considered (unions, intersections, complements) (S-CP.A.1). Students calculate probabilities based on two-way data tables and interpret them in context (S-CP.A.4). They also see how to create "hypothetical 1000" two-way tables as a way of calculating probabilities. Students are introduced to conditional probability (S-CP.A.3, S-CP.A.5), and the important concept of independence is developed (S-CP.A.2, S-CP.A.5). The final lessons in this topic introduce probability rules (S-CP.B.6, S-CP.B.7).

Topic B is a short topic consisting of four lessons. This topic introduces the idea of using a smooth curve to model a data distribution, describes properties of the normal distribution, and asks students to distinguish between data distributions for which it would be reasonable to use a normal distribution as a model and those for which a normal distribution would not be a reasonable model. In the final two lessons of this topic, students use tables and technology to find areas under a normal curve and interpret these areas in the context of modeling a data distribution (S-ID.A.4).

Topics C and D develop students' understanding of statistical inference and introduce different types of statistical studies (observational studies, surveys, and experiments) (S-IC.B.3). In Topic C, students explore using data from a random sample to estimate a population mean or a population proportion. Building on what they learned about sampling variability in Grade 7, students use simulation to create an understanding of margin of error. Students calculate the margin of error and interpret it in context (S-IC.B.4). Students also evaluate reports from the media using sample data to estimate a population mean or proportion (S-IC.B.6).

Topic D focuses on drawing conclusions based on data from a statistical experiment. Given data from a statistical experiment, students use simulation to create a randomization distribution and use it to determine if there is a significant difference between two treatments (S-IC.B.5). Students also critique and evaluate published reports based on statistical experiments that compare two treatments (S-IC.B.6).

The module comprises 30 lessons; 10 days are reserved for administering the Mid- and End-of-Module Assessments, returning the assessments, and remediating or providing further applications of the concepts. The Mid-Module Assessment follows Topic B. The End-of-Module Assessment follows Topic D.

FOCUS STANDARDS

Summarize, represent, and interpret data on a single count or measurement variable.

S-ID.A.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

Understand and evaluate random processes underlying statistical experiments.

S-IC.A.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.

S-IC.A.2 Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability* 0.5. *Would a result of* 5 *tails in a row cause you to question the model?*

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

S-IC.B.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.

S-IC.B.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.

S-IC.B.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.

S-IC.B.6 Evaluate reports based on data.

Understand independence and conditional probability and use them to interpret data.

S-CP.A.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").

S-CP.A.2 Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.

S-CP.A.3 Understand the conditional probability of A given B as P(A and B)/P(B), and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.

S-CP.A.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*

S-CP.A.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

S-CP.B.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.

S-CP.B.7 Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model.

FOUNDATIONAL STANDARDS

Use random sampling to draw inferences about a population.

7.SP.A.1 Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

7.SP.A.2 Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. *For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.*

Draw informal comparative inferences about two populations.

7.SP.B.3 Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. *For example, the mean height of players on the basketball team is* 10 cm *greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.*

7.SP.B.4 Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. *For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.*

Investigate chance processes and develop, use, and evaluate probability models.

7.SP.C.8 Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.

a. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.
b. Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., “rolling double sixes”), identify the outcomes in the sample space which compose the event.
c. Design and use a simulation to generate frequencies for compound events. *For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?*

Summarize, represent, and interpret data on a single count or measurement variable.

S-ID.A.2 Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.

S-ID.A.3 Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

Summarize, represent, and interpret data on two categorical and quantitative variables.

S-ID.B.5 Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.

FOCUS STANDARDS FOR MATHEMATICAL PRACTICE

MP.2 *Reason abstractly and quantitatively.* Students use data from a sample to estimate a population mean or proportion and generalize from a sample to the population. They associate a margin of error with estimates based on a sample and interpret them in the context of generalizing from a sample to the population. Students also make conjectures or claims about independence and use arguments based on probabilities to support them.

MP.3 *Construct viable arguments and critique the reasoning of others.* Students test conjectures about treatment differences in the context of a statistical experiment. Students critique and evaluate reports based on data from random samples and reports based on data from experiments. Students frequently develop conjectures and use statistical reasoning to evaluate them.

MP.4 *Model with mathematics.* Students use smooth curves to model data distributions. Students use the normal distribution as a model in order to answer questions about a data distribution. Students use probability models to describe real-world contexts.

MP.5 *Use appropriate tools strategically.* Students use technology to carry out simulations in order to study sampling variability. Students also use technology to compute estimates of population characteristics (such as the mean and standard deviation) and to calculate margin of error. Students use simulation to investigate statistical significance in the context of comparing treatments in a statistical experiment.

MODULE TOPIC SUMMARIES

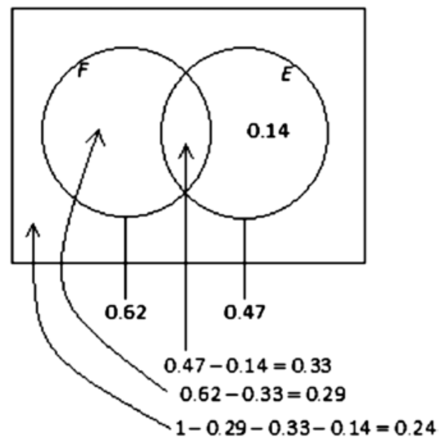
Topic A: Probability

Fundamental ideas from Grade 7 are revisited and extended to allow students to build a more formal understanding of probability. Students expand their understanding of chance experiments, sample space, and events to the more complex understanding of events defined as unions, intersections, and complements (S-CP.A.1). Students develop this understanding as they consider events that can be described as unions and intersections in the context of a game involving cards and spinners. One such game is introduced in Lesson 1, and then students explore further variations of the game in the lesson's problem set. Students also consider whether observations from a chance experiment are consistent with a given probability model (S-IC.A.2).

Students calculate probabilities of unions and intersections using data in two-way data tables and interpret them in context (S-CP.A.4). Students deepen their understanding by creating *hypothetical* 1000 two-way tables (that is, tables based on a hypothetical population of 1,000 observations) and then use these tables to calculate probabilities. Students use given probability information to determine the marginal totals and individual cell counts. This table then allows students to calculate conditional probabilities, as well as probabilities of unions, intersections, and complements, without the need for formal probability rules.

Students are introduced to conditional probability (S-CP.A.3, S-CP.A.5), which is used to illustrate the important concept of independence by describing two events, A and B, as independent if the conditional probability of A given B is not equal to the unconditional probability of A. In this case, knowing that event B has occurred does not change the assessment of the probability that event A has also occurred (S-CP.A.2, S-CP.A.5). Students use two-way tables to determine if two events are independent by calculating and interpreting conditional probabilities. In Lesson 3, students are presented with athletic participation data from Rufus King High School in two-way frequency tables, and conditional probabilities are calculated using column or row summaries. Students then use the conditional probabilities to investigate whether or not there is a connection between two events.

Students are also introduced to Venn diagrams to represent the sample space and various events. Students see how the regions of a Venn diagram connect to the cells of a two-way table. Venn diagrams also help students understand probability formulas involving the formal symbols of union, intersection, and complement.



In addition, a Venn diagram can show how subtracting the probability of an event from 1 enables students to acquire the probability of the complement of the event and why the probability of the intersection of two events is subtracted from the sum of event probabilities when calculating the probability of the union of two events.

The final lessons in this topic introduce probability rules (the multiplication rule for independent events, the addition rule for the union of two events, and the complement rule for the complement of an event) (S-CP.B.6, S-CP.B.7). Students use the multiplication rule for independent events to calculate the probability of the intersection of two events. Students interpret independence based on the conditional probability and its connection to the multiplication rule.

Focus Standards:	S-IC.A.2	Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability* 0.5. *Would a result of* 5 *tails in a row cause you to question the model?*
	S-CP.A.1	Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").
	S-CP.A.2	Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent.
	S-CP.A.3	Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*.
	S-CP.A.4	Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*
	S-CP.A.5	Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*

	S-CP.B.6	Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model.
	S-CP.B.7	Apply the Addition Rule, *P*(*A* or *B*) = *P*(*A*) + *P*(*B*) - *P*(*A* and *B*), and interpret the answer in terms of the model.
Instructional Days:	7	

Student Outcomes

Lesson 1: Chance Experiments, Sample Spaces, and Events

- Students determine the sample space for a chance experiment.
- Given a description of a chance experiment and an event, students identify the subset of outcomes from the sample space corresponding to the complement of an event.
- Given a description of a chance experiment and two events, students identify the subset of outcomes from the sample space corresponding to the union or intersection of two events.
- Students calculate the probability of events defined in terms of unions, intersections, and complements for a simple chance experiment with equally likely outcomes.

Lesson 2: Calculating Probabilities of Events Using Two-Way Tables

- Students calculate probabilities given a two-way table of data.
- Students construct a hypothetical 1000 two-way table given probability information.
- Students interpret probabilities in context.

Lesson 3: Calculating Conditional Probabilities and Evaluating Independence Using Two-Way Tables

- Students construct a hypothetical 1000 two-way table from given probability information and use the table to calculate the probabilities of events.
- Students calculate conditional probabilities given a two-way data table or using a hypothetical 1000 two-way table.
- Students interpret probabilities, including conditional probabilities, in context.

Lesson 4: Calculating Conditional Probabilities and Evaluating Independence Using Two-Way Tables

- Students use a hypothetical 1000 two-way table to calculate probabilities of events.
- Students calculate conditional probabilities given a two-way data table or using a hypothetical 1000 two-way table.
- Students use two-way tables (data tables or hypothetical 1000 two-way tables) to determine if two events are independent.
- Students interpret probabilities, including conditional probabilities, in context.

Lesson 5: Events and Venn Diagrams

- Students represent events by shading appropriate regions in a Venn diagram.
- Given a chance experiment with equally likely outcomes, students calculate counts and probabilities by adding or subtracting given counts or probabilities.
- Students interpret probabilities in context.

Lesson 6: Probability Rules

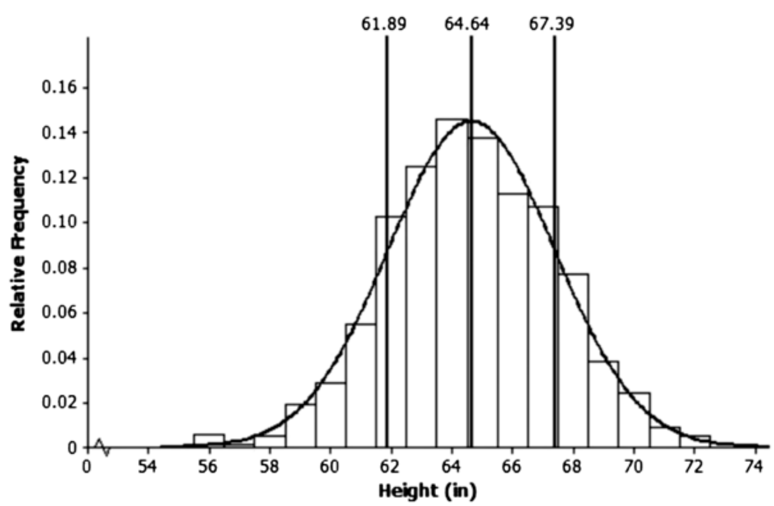
- Students use the complement rule to calculate the probability of the complement of an event and the multiplication rule for independent events to calculate the probability of the intersection of two independent events.
- Students recognize that two events A and B are independent if and only if $P(A \text{ and } B) = P(A)P(B)$ and interpret independence of two events A and B as meaning that the conditional probability of A given B is equal to $P(A)$.
- Students use the formula for conditional probability to calculate conditional probabilities and interpret probabilities in context.

Lesson 7: Probability Rules

- Students use the addition rule to calculate the probability of a union of two events.
- Students interpret probabilities in context.

Topic B: Modeling Data Distributions

This topic introduces students to the idea of using a smooth curve to model a data distribution, eventually leading to using the normal distribution to model data distributions that are bell shaped and symmetric. Many naturally occurring variables, such as arm span, weight, reaction times, and standardized test scores, have distributions that are well described by a normal curve.



Students begin by reviewing their previous work with shape, center, and variability. Students use the mean and standard deviation to describe center and variability for a data distribution that is approximately symmetric. This provides a foundation for selecting an appropriate normal distribution to model a given data distribution.

Students learn to draw a smooth curve that could be used to model a given data distribution. A smooth curve is first used to model a relative frequency histogram, which shows that the area under the curve represents the approximate proportion of data falling in a given interval. Properties of the normal distribution are introduced by asking students to recognize when it is reasonable and when it is unreasonable to use a normal distribution model for a given data distribution. Students use tables and technology to calculate normal probabilities. They work with graphing calculators, tables of normal curve areas, and spreadsheets to calculate probabilities in the examples and exercises provided (S-ID.A.4).

Focus Standard:	S-ID.A.4	Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.
Instructional Days:	4	

Student Outcomes

Lesson 8: Distributions—Center, Shape, and Spread

- Students describe data distributions in terms of shape, center, and variability.
- Students use the mean and standard deviation to describe center and variability for a data distribution that is approximately symmetric.

Lesson 9: Using a Curve to Model a Data Distribution

- Students draw a smooth curve that could be used as a model for a given data distribution.
- Students recognize when it is reasonable and when it is not reasonable to use a normal curve as a model for a given data distribution.

Lesson 10: Normal Distributions

- Students calculate z-scores.
- Students use technology and tables to estimate the area under a normal curve.
- Students interpret probabilities in context.

Lesson 11: Normal Distributions

- Students use tables and technology to estimate the area under a normal curve.
- Students interpret probabilities in context.
- When appropriate, students select an appropriate normal distribution to serve as a model for a given data distribution.

Topic C: Drawing Conclusions Using Data from a Sample

This topic introduces different types of statistical studies (e.g., observational studies, surveys, and experiments) (S-IC.B.3). The role of randomization (i.e., random selection in observational studies and surveys and random assignment in experiments) is addressed. A discussion of random selection (i.e., selecting a sample at random from a population of interest) shows students how selecting participants at random provides a representative sample, thereby allowing conclusions to be generalized from the sample to the population. A discussion of random assignment in experiments, which involves assigning subjects to experimental groups at random, helps students see that random assignment is designed to create comparable groups, making it possible to assess the effects of an explanatory variable on a response.

The distinction between population characteristics and sample statistics (first made in Grade 7) is revisited. Scenarios are introduced in which students are asked a statistical question that involves estimating a population mean or a population proportion. For example, students are asked to define an appropriate population, a population characteristic, a sample, and sample statistics that might be used in a study of the time it takes students to run a quarter mile or a study of the proportion of national parks that contain bald eagle nests.

In this topic, students use data from a random sample to estimate a population mean or a population proportion. Building on what they learned about sampling variability in Grade 7, students use simulation to create an understanding of margin of error. In Grade 7, students learned that the proportion of successes in a random sample from a population varies from sample to sample due to the random selection process. They understand that the value of the sample proportion is not exactly equal to the value of the population proportion. In Algebra II, they use margin of error to describe how different the value of the sample proportion might be from the value of the population proportion. Students begin by using a physical simulation process to carry out a simulation. Starting with a population that contains 40% successes (using a bag with 40 black beans and 60 white beans), they select random samples from the population and calculate the sample proportion. By doing this many times, they are able to get a sense of what kind of differences are likely. Their understanding should then extend to include the concept of margin of error. Students then proceed to use technology to carry out a simulation. Once students understand the concept of margin of error, they go on to learn how to calculate and interpret it in context (S-IC.A.1, S-IC.B.4). Students also evaluate reports from the media in which sample data are used to estimate a population mean or proportion (S-IC.B.6).

Focus Standards:	S-IC.A.1	Understand statistics as a process for making inferences about population parameters based on a random sample from that population.
	S-IC.B.3	Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.
	S-IC.B.4	Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.
	S-IC.B.6	Evaluate reports based on data.
Instructional Days:	11	

Student Outcomes

Lesson 12: Types of Statistical Studies

- Students distinguish between observational studies, surveys, and experiments.
- Students explain why random selection is an important consideration in observational studies and surveys and why random assignment is an important consideration in experiments.
- Students recognize when it is reasonable to generalize the results of an observational study or survey to some larger population and when it is reasonable to reach a cause-and-effect conclusion about the relationship between two variables.

Lesson 13: Using Sample Data to Estimate a Population Characteristic

- Students differentiate between a population and a sample.
- Students differentiate between a population characteristic and a sample statistic.
- Students recognize statistical questions that are answered by estimating a population mean or a population proportion.

Lesson 14: Sampling Variability in the Sample Proportion

- Students understand the term *sampling variability* in the context of estimating a population proportion.
- Students understand that the standard deviation of the sampling distribution of the sample proportion offers insight into the accuracy of the sample proportion as an estimate of the population proportion.

Lesson 15: Sampling Variability in the Sample Proportion

- Students understand the term *sampling variability* in the context of estimating a population proportion.
- Students understand that the standard deviation of the sampling distribution of the sample proportion offers insight into the accuracy of the sample proportion as an estimate of the population proportion.

Lesson 16: Margin of Error When Estimating a Population Proportion

- Students use data from a random sample to estimate a population proportion.
- Students calculate and interpret margin of error in context.
- Students know the relationship between sample size and margin of error in the context of estimating a population proportion.

Lesson 17: Margin of Error When Estimating a Population Proportion

- Students use data from a random sample to estimate a population proportion.
- Students calculate and interpret margin of error in context.
- Students know the relationship between sample size and margin of error in the context of estimating a population proportion.

Lesson 18: Sampling Variability in the Sample Mean

- Students understand the term *sampling variability* in the context of estimating a population mean.
- Students understand that the standard deviation of the sampling distribution of the sample mean offers insight into the accuracy of the sample mean as an estimate of the population mean.

Lesson 19: Sampling Variability in the Sample Mean

- Students understand the term *sampling variability* in the context of estimating a population mean.
- Students understand that the standard deviation of the sampling distribution of the sample mean conveys information about the anticipated accuracy of the sample mean as an estimate of the population mean.

Lesson 20: Margin of Error When Estimating a Population Mean

- Students use data from a random sample to estimate a population mean.
- Students calculate and interpret margin of error in context.
- Students know the relationship between sample size and margin of error in the context of estimating a population mean.

Lesson 21: Margin of Error When Estimating a Population Mean

- Students use data from a random sample to estimate a population mean.
- Students calculate and interpret margin of error in context.
- Students know the relationship between sample size and margin of error in the context of estimating a population mean.

Lesson 22: Evaluating Reports Based on Data from a Sample

- Students interpret margin of error from reports that appear in newspapers and other media.
- Students critique and evaluate statements in published reports that involve estimating a population proportion or a population mean.

Topic D: Drawing Conclusions Using Data from an Experiment

This topic focuses on drawing conclusions based on data from a statistical experiment. Experiments are introduced as investigations designed to compare the effect of two treatments on a response variable. Students revisit the distinction between random selection and random assignment.

When comparing two treatments using data from a statistical experiment, it is important to assess whether the observed difference in group means indicates a real difference between the treatments in the experiment or whether it is possible that there is no difference and that the observed difference is just a by-product of the random assignment of subjects to treatments (S-IC.B.5). To help students understand how this distinction is made, lessons in this topic use simulation to create a randomization distribution as a way of exploring the types of differences they might expect to see by chance when there is no real difference between groups. By understanding these differences, students are able to determine whether an observed difference in means is significant (S-IC.B.5).

Students also critique and evaluate published reports based on statistical experiments that compare two treatments (S-IC.B.6). For example, students read a short summary of an article in the online *New England Journal of Medicine* describing an experiment to determine if wearing a brace helps adolescents with scoliosis. Then they watch an online video report for the *Wall Street Journal* titled “BMW Drivers Really Are Jerks” that describes a study of the relationship between driving behavior and the type of car driven.

Focus Standards:	S-IC.B.3	Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.
	S-IC.B.5	Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.
	S-IC.B.6	Evaluate reports based on data.
Instructional Days:	8	

Student Outcomes

Lesson 23: Experiments and the Role of Random Assignment

- Given a description of a statistical experiment, students identify the response variable and the treatments.
- Students recognize the different purposes of random selection and of random assignment.
- Students recognize the importance of random assignment in statistical experiments.

Lesson 24: Differences Due to Random Assignment Alone

- Students understand that when one group is randomly divided into two groups, the two groups’ means differ just by chance (a consequence of the random division).

- Students understand that when one group is randomly divided into two groups, the distribution of the difference in the two groups' means can be described in terms of shape, center, and spread.

Lesson 25: Ruling Out Chance

- Given data from a statistical experiment with two treatments, students create a randomization distribution.
- Students use a randomization distribution to determine if there is a significant difference between two treatments.

Lesson 26: Ruling Out Chance

- Given data from a statistical experiment with two treatments, students create a randomization distribution.
- Students use a randomization distribution to determine if there is a significant difference between two treatments.

Lesson 27: Ruling Out Chance

- Given data from a statistical experiment with two treatments, students create a randomization distribution.
- Students use a randomization distribution to determine if there is a significant difference between two treatments.

Lesson 28: Drawing a Conclusion from an Experiment

- Students carry out a statistical experiment to compare two treatments.
- Given data from a statistical experiment with two treatments, students create a randomization distribution.
- Students use a randomization distribution to determine if there is a significant difference between two treatments.

Lesson 29: Drawing a Conclusion from an Experiment

- Students carry out a statistical experiment to compare two treatments.
- Given data from a statistical experiment with two treatments, students create a randomization distribution.
- Students use a randomization distribution to determine if there is a significant difference between two treatments.

Lesson 30: Evaluating Reports Based on Data from an Experiment

- Students critique and evaluate statements in published reports that involve determining if there is a significant difference between two treatments in a statistical experiment.

CHAPTER 7

Terminology

The terms included in this index were compiled from the New or Recently Introduced Terms portion of the Terminology section of the Module Overviews in A *Story of Functions*. All four courses are represented, from Algebra I to Precalculus. This index serves as a reference for teachers to quickly determine at which point in the curriculum terms are introduced.

ALGEBRA I

Module 1

- **Algebraic Expression** An *algebraic expression* is either: (1) a numerical symbol or a variable symbol or (2) the result of placing previously generated algebraic expressions into the two blanks of one of the four operators ((__) + (__), (__) − (__), (__) × (__), (__) ÷ (__)) or into the base blank of an exponentiation with an exponent that is a rational number.
- **Constant Term of a Polynomial in Standard Form** The *constant term* is the value of the numerical expression found by substituting 0 into all the variable symbols of the polynomial, namely a_0.
- **Degree of a Monomial** The *degree of a nonzero monomial* is the sum of the exponents of the variable symbols that appear in the monomial.
- **Degree of a Polynomial in Standard Form** The *degree of a polynomial in standard form* is the highest degree of the terms in the polynomial, namely n.
- **Equivalent Algebraic Expressions** Two *algebraic expressions* are *equivalent* if we can convert one expression into the other by repeatedly applying the commutative, associative, and distributive properties and the properties of rational exponents to components of the first expression.
- **Equivalent Numerical Expressions** Two *numerical expressions* are *equivalent* if they evaluate to the same number.

- **Graph of an Equation in Two Variables** The set of all points in the coordinate plane that are solutions to an equation in two variables is called the *graph of the equation*.
- **Leading Term and Leading Coefficient of a Polynomial in Standard Form** The term $a_n x^n$ is called the *leading term*, and a_n is called the *leading coefficient*.
- **Monomial** A *monomial* is a polynomial expression generated using only the multiplication operator (__ × __). Monomials are products whose factors are numerical expressions or variable symbols.
- **Numerical Expression** A *numerical expression* is an algebraic expression that contains only numerical symbols (no variable symbols) and that evaluates to a single number.
- **Numerical Symbol** A *numerical symbol* is a symbol that represents a specific number.
- **Piecewise Linear Function** Given a finite number of non-overlapping intervals on the real number line, a (real) *piecewise linear function* is a function from the union of the intervals to the set of real numbers such that the function is defined by (possibly different) linear functions on each interval.
- **Polynomial Expression** A *polynomial expression* is either: (1) a numerical expression or a variable symbol or (2) the result of placing two previously generated polynomial expressions into the blanks of the addition operator (__ + __) or the multiplication operator (__ × __).
- **Solution** A *solution* to an equation with one variable is a number in the domain of the variable that, when substituted for all instances of the variable in both expressions, makes the equation a true number sentence.
- **Solution Set** The set of solutions of an equation is called its *solution set*.
- **Standard Form of a Polynomial Expression in One Variable** A polynomial expression with one variable symbol x is in *standard form* if it is expressed as $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where n is a nonnegative integer, and $a_0, a_1, a_2, \ldots, a_n$ are constant coefficients with $a_n \neq 0$. A polynomial expression in x that is in standard form is often called a *polynomial in x*.
- **Variable Symbol** A *variable symbol* is a symbol that is a placeholder for a number. It is possible that a question may restrict the type of number that a placeholder might permit, maybe integers only or a positive real number, for instance.
- **Zero Product Property** The *Zero Product Property* states that given real numbers a and b, if $a \cdot b = 0$ then either $a = 0$ or $b = 0$, or both a and $b = 0$.

Module 2

- **Association** A statistical *association* is any relationship between measures of two types of quantities so that one is statistically dependent on the other.
- **Conditional Relative Frequency** A *conditional relative frequency* compares a frequency count to the marginal total that represents the condition of interest.
- **Correlation Coefficient** The *correlation coefficient*, often denoted by r, is a number between –1 and +1, inclusively, that measures the strength and direction of a linear relationship between the two types of quantities. If $r = 1$ or $r = -1$, then the graph of data points of the bivariate data set lie on a line of positive or negative slope.

- **Interquartile Range** The *interquartile range* (or *IQR*) is the distance between the first quartile and the third quartile: $IQR = Q3 - Q1$. The IQR describes variability by identifying the length of the interval that contains the middle 50% of the data values.
- **Outlier** An *outlier* of a finite numerical data set is a value that is greater than Q3 by a distance of $1.5 \cdot IQR$, or a value that is less than Q1 by a distance of $1.5 \cdot IQR$. Outliers are usually identified by an "*" or a "•" in a box plot.
- **Residual** The *residual* of the data point (x_i, y_i) is the (actual y_i-value) – (predicted y-value) for the given x_i.
- **Residual Plot** Given a bivariate data set and linear equation used to model the data set, a *residual plot* is the graph of all ordered pairs determined as follows: For each data point (x_i, y_i) in the data set, the first entry of the ordered pair is the x-value of the data point, and the second entry is the residual of the data point.
- **Sample Standard Deviation** The *sample variance* for a numerical sample data set of n-values is the sum of the squared distances the values are from the mean divided by $(n-1)$. The *sample standard deviation* is the principle (positive) square root of the sample variance.
- **Skewed Data Distribution** A data distribution is said to be *skewed* if the distribution is not symmetric with respect to its mean. Left-skewed or skewed to the left is indicated by the data spreading out longer (like a tail) on the left side. Right-skewed or skewed to the right is indicated by the data spreading out longer (like a tail) on the right side.

Module 3

- **Average Rate of Change** Given a function f whose domain includes the closed interval of real numbers $[a, b]$ and whose range is a subset of the real numbers, the *average rate of change* on the interval $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$.
- **Domain** Refer to the definition of *function*.
- **Function** A *function* is a correspondence between two sets, X and Y, in which each element of X is matched[1] to one and only one element of Y. The set X is called the *domain*; the set Y is called the *range*.
- **Linear Function** A *linear function* is a polynomial function of degree 1.
- **Piecewise Linear Function** Given non-overlapping intervals on the real number line, a (real) *piecewise linear function* is a function from the union of the intervals to the set of real numbers such that the function is defined by (possibly different) linear functions on each interval.
- **Range** Refer to the definition of *function*.

Module 4

- **Axis of Symmetry of the Graph of a Quadratic Function** Given a quadratic function in standard form, $f(x) = ax^2 + bx + c$, the vertical line given by the graph of the equation, $x = -\frac{b}{2a}$, is called the *axis of symmetry of the graph of the quadratic function*.
- **Cube Root Function** The parent function $f(x) = \sqrt[3]{x}$.
- **Cubic Function** A polynomial function of degree 3.

- **Degree of a Monomial Term** The *degree of a monomial term* is the sum of the exponents of the variables that appear in a term of a polynomial.
- **Degree of a Polynomial** The *degree of a polynomial* in one variable in standard form is the highest degree of the terms in the polynomial.
- **Discriminant** The *discriminant* of a quadratic function in the form $ax^2 + bx + c = 0$ is $b^2 - 4ac$. The nature of the roots of a quadratic equation can be identified by determining if the discriminant is positive, negative, or equal to zero.
- **End Behavior of a Quadratic Function** Given a quadratic function in the form $f(x) = ax^2 + bx + c$ (or $f(x) = a(x-h)^2 + k$), the quadratic function is said to *open up* if $a > 0$ and *open down* if $a < 0$.
- **Factored Form for a Quadratic Function** A quadratic function written in the form $f(x) = a(x-n)(x-m)$.
- **Leading Coefficient** The *leading coefficient* of a polynomial is the coefficient of the term of highest degree.
- **Parent Function** A *parent function* is the simplest function in a "family" of functions that can each be formed by one or more transformations of another.
- **Quadratic Formula** The *quadratic formula* is the formula that emerges from solving the general form of a quadratic equation by completing the square, $y = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. It can be used to solve any quadratic equation.
- **Quadratic Function** A polynomial function of degree 2.
- **Roots of a Polynomial Function** The domain values for a polynomial function that make the value of the polynomial function equal zero when substituted for the variable.
- **Square Root Function** The parent function $f(x) = \sqrt{x}$.
- **Standard Form for a Quadratic Function** A quadratic function written in the form $f(x) = ax^2 + bx + c$.
- **Standard Form of a Polynomial in One Variable** A polynomial expression with one variable symbol x is in *standard form* if it is expressed as $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where n is a non-negative integer, and $a_0, a_1, a_2, \ldots, a_n$ are constant coefficients with $a_n \neq 0$.
- **Vertex Form** Completed-square form for a quadratic function; in other words, written in the form $f(x) = a(x-h)^2 + k$.
- **Vertex of the Graph of a Quadratic Function** The point where the graph of a quadratic function and its axis of symmetry intersect is called the vertex. The vertex is either a maximum or a minimum of the quadratic function, depending on whether the leading coefficient of the function in standard form is negative or positive, respectively.

***Module* 5**

- **Analytic Model** An *analytic model* is one that seeks to explain data based on deeper theoretical ideas—for example, by using an algebraic equation. This is sometimes referred to as a symbolic model.
- **Descriptive Model** A *descriptive model* is one that seeks to describe phenomena or summarize them in a compact form—for example, by using a graph.

GEOMETRY

Module 1

- **Isometry** An *isometry* of the plane is a transformation of the plane that is distance preserving.

Module 2

- **Cosine** Let θ be the angle measure of an acute angle of the right triangle. The *cosine of θ of a right triangle* is the value of the ratio of the length of the adjacent side (denoted *adj*) to the length of the hypotenuse (denoted *hyp*). As a formula, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$.
- **Dilation** For $r > 0$, a *dilation with center C and scale factor r* is a transformation $D_{C,r}$ of the plane defined as follows:
  1. For the center C, $D_{C,r}(C) = C$, and
  2. For any other point P, $D_{C,r}(P)$ is the point Q on $\overrightarrow{CP}$ so that $CQ = r \cdot CP$.
- **Sides of a Right Triangle** The *hypotenuse* of a right triangle is the side opposite the right angle; the other two sides of the right triangle are called the *legs*. Let θ be the angle measure of an acute angle of the right triangle. The *opposite side* is the leg opposite that angle. The *adjacent side* is the leg that is contained in one of the two rays of that angle (the hypotenuse is contained in the other ray of the angle).
- **Similar** Two figures in a plane are *similar* if there exists a similarity transformation taking one figure onto the other figure. A congruence is a similarity with scale factor 1. It can be shown that a similarity with scale factor 1 is a congruence.
- **Similarity Transformation** A *similarity transformation* (or *similarity*) is a composition of a finite number of dilations or basic rigid motions. The *scale factor* of a similarity transformation is the product of the scale factors of the dilations in the composition; if there are no dilations in the composition, the scale factor is defined to be 1. A similarity is an example of a transformation.
- **Sine** Let θ be the angle measure of an acute angle of the right triangle. The *sine of θ of a right triangle* is the value of the ratio of the length of the opposite side (denoted *opp*) to the length of the hypotenuse (denoted *hyp*). As a formula, $\sin\theta = \frac{\text{opp}}{\text{hyp}}$.
- **Tangent** Let θ be the angle measure of an acute angle of the right triangle. The *tangent of θ of a right triangle* is the value of the ratio of the length of the opposite side (denoted *opp*) to the length of the adjacent side (denoted *adj*). As a formula, $\tan\theta = \frac{\text{opp}}{\text{adj}}$.

  Note that in Algebra II, sine, cosine, and tangent are thought of as functions whose domains are subsets of the real numbers; they are not considered as values of ratios. Thus, in Algebra II, the values of these functions for a given θ are notated as $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ using function notation (i.e., parentheses are included).

Module 3

- **Cavalieri's Principle** Given two solids that are included between two parallel planes, if every plane parallel to the two planes intersects both solids in cross-sections of equal area, then the volumes of the two solids are equal.

- **Cone** Let B be a region in a plane E, and V be a point not in E. The *cone with base B and vertex* V is the union of all segments VP for all points P in B. If the base is a polygonal region, then the cone is usually called a *pyramid.*
- **General Cylinder** Let E and E′ be two parallel planes, let B be a region in the plane E, and let L be a line which intersects E and E′ but not B. At each point P of B, consider the segment $\overline{PP'}$ parallel to L, joining P to a point P′ of the plane E′. The union of all these segments is called a *cylinder with base* B.
- **Inscribed Polygon** A polygon is *inscribed in* a circle if all of the vertices of the polygon lie on the circle.
- **Intersection** The *intersection* of A and B is the set of all objects that are elements of A and also elements of B. The intersection is denoted $A \cap B$.
- **Rectangular Pyramid** Given a rectangular region B in a plane E, and a point V not in E, the *rectangular pyramid with base B and vertex* V is the union of all segments VP for points P in B.
- **Right Rectangular Prism** Let E and E′ be two parallel planes. Let B be a rectangular region in the plane E. At each point P of B, consider the segment PP′ perpendicular to E, joining P to a point P′ of the plane E′. The union of all these segments is called a *right rectangular prism.*
- **Solid Sphere or Ball** Given a point C in the three-dimensional space and a number $r > 0$, the *solid sphere (or ball) with center C and radius r* is the set of all points in space whose distance from point C is less than or equal to r.
- **Sphere** Given a point C in the three-dimensional space and a number $r > 0$, the *sphere with center C and radius r* is the set of all points in space that are distance r from the point C.
- **Subset** A set A is a *subset* of a set B if every element of A is also an element of B.
- **Tangent to a Circle** A *tangent line to a circle* is a line that intersects a circle in one and only one point.
- **Union** The *union* of A and B is the set of all objects that are either elements of A or of B or of both. The union is denoted $A \cup B$.

***Module* 4**

- **Normal Segment to a Line** A line segment with one endpoint on a line and perpendicular to the line is called a *normal segment* to the line.

***Module* 5**

- **Arc Length** The *length of an arc* is the circular distance around the arc.
- **Central Angle** A *central angle* of a circle is an angle whose vertex is the center of a circle.
- **Chord** Given a circle C, let P and Q be points on C. Then $\overline{PQ}$ is called a *chord* of C.
- **Cyclic Quadrilateral** A quadrilateral inscribed in a circle is called a *cyclic quadrilateral.*
- **Inscribed Angle** An *inscribed angle* is an angle whose vertex is on a circle, and each side of the angle intersects the circle in another point.

- **Inscribed Polygon** A polygon is *inscribed* in a circle if all vertices of the polygon lie on the circle.
- **Secant Line** A *secant line* to a circle is a line that intersects a circle in exactly two points.
- **Sector** Let AB be an arc of a circle. The *sector* of a circle with arc AB is the union of all radii of the circle that have an endpoint in arc AB. The arc AB is called the *arc of the sector*, and the length of any radius of the circle is called the *radius of the sector*.
- **Tangent Line** A *tangent line* to a circle is a line in the same plane that intersects the circle in one and only one point. This point is called the *point of tangency*.

ALGEBRA II

Module 1

- **Axis of Symmetry** The *axis of symmetry of a parabola* given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus.
- **Dilation at the Origin** A dilation at the origin D_k is a horizontal scaling by $k > 0$ followed by a vertical scaling by the same factor k. In other words, this dilation of the graph of $y = f(x)$ is the graph of the equation $y = kf(\frac{1}{k}x)$. A *dilation at the origin* is a special type of a dilation.
- **End Behavior** Let f be a function whose domain and range are subsets of the real numbers. The *end behavior* of a function f is a description of what happens to the values of the function
    - as x approaches positive infinity and
    - as x approaches negative infinity.
- **Even Function** Let f be a function whose domain and range is a subset of the real numbers. The function f is called *even* if the equation $f(x) = f(-x)$ is true for every number x in the domain. Even-degree polynomial functions are sometimes even functions, such as $f(x) = x^{10}$, and sometimes not, such as $g(x) = x^2 - x$.
- **Odd Function** Let f be a function whose domain and range is a subset of the real numbers. The function f is called *odd* if the equation $f(-x) = -f(x)$ is true for every number x in the domain. Odd-degree polynomial functions are sometimes odd functions, such as $f(x) = x^{11}$, and sometimes not, such as $h(x) = x^3 - x^2$.
- **Parabola** A *parabola* with *directrix line* L and *focus point* F is the set of all points in the plane that are equidistant from the point F and line L.
- **Pythagorean Triple** A *Pythagorean triple* is a triplet of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. The triple (3, 4, 5) is a Pythagorean triple, but $\left(1, 1, \sqrt{2}\right)$ is not, even though the numbers are side lengths of an isosceles right triangle.
- **Rational Expression** A *rational expression* is either a numerical expression or a variable symbol or the result of placing two previously generated rational expressions into the blanks of the addition operator (__ + __), the subtraction operator (__ – __), the multiplication operator (__ × __), or the division operator (__ ÷ __).

- **A Square Root of a Number** A *square root of a number* x is a number whose square is x. In symbols, a square root of x is a number a such that $a^2 = x$. Negative numbers do not have any real square roots, zero has exactly one real square root, and positive numbers have two real square roots.
- **The Square Root of a Number** Every positive real number x has a unique positive square root called *the square root* or *principal square root* of x; it is denoted $\sqrt{x}$. The square root of zero is zero.
- **Vertex of a Parabola** The *vertex of a parabola* is the point where the axis of symmetry intersects the parabola.

Module 2

- **Amplitude** The *amplitude* is the distance between a maximal point of the graph of the sinusoidal function and the midline.
- **Cosecant** Let θ be any real number such that $\theta \neq k\pi$ for all integers k. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\csc(\theta)$ is $\frac{1}{y_\theta}$.
- **Cosine** Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\cos(\theta)$ is x_θ.
- **Cotangent** Let θ be any real number such that $\theta \neq k\pi$ for all integers k. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\cot(\theta)$ is $\frac{x_\theta}{y_\theta}$.
- **Frequency** The *frequency* of a periodic function is the unit rate of the constant rate defined by the number of cycles per unit length.
- **Midline** The *midline* is the horizontal line that is halfway between the maximal line and the minimal line.
- **Period** The *period* P is the distance between two consecutive maximal points or two consecutive minimal points on the graph of a sinusoidal function.
- **Periodic Function** A function f whose domain is a subset of the real numbers is said to be *periodic* with period $P > 0$ if the domain of f contains $x + P$ whenever it contains x, and if $f(x + P) = f(x)$ for all real numbers x in its domain.
- **Radian** A *radian angle* is the angle subtended by an arc of a circle that is equal in length to the radius of the circle. A *radian* (1 rad) is a unit of rotational measure given by a rotation by a radian angle.
- **Secant** Let θ be any real number such that $\theta \neq \frac{\pi}{2} + k\pi$ for all integers k. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\sec(\theta)$ is $\frac{1}{x_\theta}$.
- **Sine** Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\sin(\theta)$ is y_θ.

- **Sinusoidal Function** A periodic function is *sinusoidal* if it can be written in the form $f(x) = A\sin(\omega(x-h)) + k$ for real numbers A, ω, h, and k. In this form,
  - $|A|$ is called the *amplitude* of the function,
  - $\frac{2\pi}{|\omega|}$ is the *period* of the function,
  - $\frac{|\omega|}{2\pi}$ is the *frequency* of the function,
  - h is called the *phase shift*, and
  - the graph of $y = k$ is called the *midline*.

  Furthermore, we can see that the graph of the sinusoidal function f is obtained by first vertically scaling the graph of the sine function by A, then horizontally scaling the resulting graph by $\frac{1}{\omega}$, and, finally, by horizontally and vertically translating the resulting graph by h and k units, respectively.
- **Tangent** Let θ be any real number such that $\theta \neq \frac{\pi}{2} + k\pi$ for all integers k. In the Cartesian plane, rotate the initial ray by θ radians about the origin.

  Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\tan(\theta)$ is $\frac{y_\theta}{x_\theta}$.
- **Trigonometric Identity** A *trigonometric identity* is a statement that two trigonometric functions are equivalent.

Module 3

- ***e*** Euler's number, e, is an irrational number that is approximately equal to 2.7182818284590.
- **Σ** The Greek letter sigma, Σ, is used to represent the sum. There is no rigid way to use Σ to represent a summation, but all notations generally follow the same rules. The most common way it is used is discussed. Given the sequence $a_1, a_2, a_3, a_4, \ldots$, we can write the sum of the first n terms of the sequence using the expression:

$$\sum_{k=1}^{n} a_k.$$

- **Arithmetic Series** An *arithmetic series* is a series whose terms form an arithmetic sequence.
- **Geometric Series** A *geometric series* is a series whose terms form a geometric sequence.
- **Invertible Function** Let f be a function whose domain is the set X and whose image is the set Y. Then f is *invertible* if there exists a function g with domain Y and image X such that f and g satisfy the property:

  For all $x \in X$ and $y \in Y$, $f(x) = y$ if and only if $g(y) = x$.

  The function g is called the *inverse* of f and is denoted f^{-1}.

  The way to interpret the property is to look at all pairs $(x, y) \in X \times Y$: If the pair (x, y) makes $f(x) = y$ a true equation, then $g(y) = x$ is a true equation. If it makes $f(x) = y$ a false equation, then $g(y) = x$ is false. If that happens for each pair in $X \times Y$, then f and g are invertible and are inverses of each other.

- **Logarithm** If three numbers L, b, and x are related by $x = b^L$, then L is the *logarithm base* b *of* x, and we write $L = \log_b(x)$. That is, the value of the expression $\log_b(x)$ is the power of b needed to be equivalent to x.

  Valid values of b as a base for a logarithm are $0 < b < 1$ and $b > 1$.
- **Series** Let $a_1, a_2, a_3, a_4, \ldots$ be a sequence of numbers. A sum of the form

  $$a_1 + a_2 + a_3 + \cdots + a_n$$

  for some positive integer n is called a *series*, or *finite series*, and is denoted S_n. The a_i's are called the *terms* of the series. The number that the series adds to is called the *sum* of the series. Sometimes S_n is called the nth *partial sum*.

Module 4

- **Complement of an Event** The *complement of an event*, A, denoted by A^C, is the event that A does not occur.
- **Conditional Probability** The probability of an event given that some other event occurs. The *conditional probability* of A given B is denoted by $P(A \mid B)$.
- **Experiment** An *experiment* is a study in which subjects are assigned to treatments for the purpose of seeing what effect the treatments have on some response.
- **Hypothetical 1000 Table** A *hypothetical 1000 table* is a two-way table that is constructed using given probability information. It represents a hypothetical population of 1,000 individuals that is consistent with the given probability distribution and also allows calculation of other probabilities of interest.
- **Independent Events** Two events A and B are independent if $P(A \mid B) = P(A)$. This implies that knowing that B has occurred does not change the probability that A has occurred.
- **Intersection of Two Events** The *intersection of two events* A and B, denoted by $A \cap B$, is the event that A and B both occur.
- **Lurking Variable** A *lurking variable* is one that causes two variables to have a high relationship even though there is no real direct relationship between the two variables.
- **Margin of Error** The *margin of error* is the maximum likely error when data from a sample are used to estimate a population characteristic, such as a population proportion or a population mean.
- **Normal Distribution** A *normal distribution* is a distribution that is bell shaped and symmetric.
- **Observational Study** An *observational study* is one in which the values of one or more variables are observed with no attempt to affect the outcomes.
- **Random Assignment** *Random assignment* is the process of using a chance mechanism to assign individuals to treatments in an experiment.
- **Random Selection** *Random selection* is the process of selecting individuals for a sample using a chance mechanism that ensures that every individual in the population has the same chance of being selected.

- **Sample Survey** A *sample survey* is an observational study in which people respond to one or more questions.
- **Treatment** A *treatment* is something administered in an experimental study.
- **Union of Two Events** The *union of two events* A and B, denoted by $A \cup B$, is the event that either A or B or both occur.

PRECALCULUS AND ADVANCED TOPICS

Module 1

- **Argument** The *argument* of the complex number z is the radian (or degree) measure of the counterclockwise rotation of the complex plane about the origin that maps the initial ray (i.e., the ray corresponding to the positive real axis) to the ray from the origin through the complex number z in the complex plane. The argument of z is denoted $\arg(z)$.
- **Bound Vector** A *bound vector* is a directed line segment (an *arrow*). For example, the directed line segment $\overrightarrow{AB}$ is a bound vector whose initial point (or *tail*) is A and terminal point (or *tip*) is B.

  Bound vectors are *bound* to a particular location in space. A bound vector $\overrightarrow{AB}$ has a magnitude given by the length of $\overline{AB}$ and direction given by the ray $\overrightarrow{AB}$. Many times, only the magnitude and direction of a bound vector matter, not its position in space. In that case, any translation of that bound vector is considered to represent the same free vector.
- **Complex Number** A *complex number* is a number that can be represented by a point in the complex plane. A complex number can be expressed in two forms:
  1. The *rectangular form* of a complex number z is $a + bi$ where z corresponds to the point (a, b) in the complex plane, and i is the imaginary unit. The number a is called the *real part* of $a + bi$, and the number b is called the *imaginary part* of $a + bi$. Note that both the real and imaginary parts of a complex number are themselves real numbers.
  2. For $z \neq 0$, the *polar form* of a complex number z is $r(\cos(\theta) + i\sin(\theta))$ where $r = |z|$ and $\theta = \arg(z)$, and i is the imaginary unit.
- **Complex Plane** The *complex plane* is a Cartesian plane equipped with addition and multiplication operators defined on ordered pairs by the following:
  - Addition: $(a,b) + (c,d) = (a + c, b + d)$.
    When expressed in rectangular form, if $z = a + bi$ and $w = c + di$, then $z + w = (a + c) + (b + d)i$.
  - Multiplication: $(a,b) \cdot (c,d) = (ac - bd, ad + bc)$.
    When expressed in rectangular form, if $z = a + bi$ and $w = c + di$, then $z \cdot w = (ac - bd) + (ad + bc)i$. The horizontal axis corresponding to points of the form $(x, 0)$ is called the *real axis*, and a vertical axis corresponding to points of the form $(0, y)$ is called the *imaginary axis*.
- **Conjugate** The *conjugate* of a complex number of the form $a + bi$ is $a - bi$. The conjugate of z is denoted $\overline{z}$.

- **Determinant of 2×2 Matrix** The *determinant of the* 2×2 *matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number computed by evaluating $ad-bc$ and is denoted by $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$.
- **Determinant of 3×3 Matrix** The *determinant of the* 3×3 *matrix* $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the number computed by evaluating the expression
  $a_{11}\det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{12}\det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right) + a_{13}\det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right)$,
  and is denoted by $\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$.
- **Directed Graph** A *directed graph* is an ordered pair $D(\mathrm{V},\mathrm{E})$ with
  - V a set whose elements are called *vertices* or *nodes*, and
  - E a set of ordered pairs of vertices, called *arcs* or *directed edges*.
- **Directed Segment** A *directed segment* $\overline{\mathrm{AB}}$ is the line segment AB together with a direction given by connecting an initial point A to a terminal point B.
- **Free Vector** A *free vector* is the equivalence class of all directed line segments (*arrows*) that are equivalent to each other by translation. For example, scientists often use free vectors to describe physical quantities that have magnitude and direction only, *freely* placing an arrow with the given magnitude and direction anywhere in a diagram where it is needed. For any directed line segment in the equivalence class defining a free vector, the directed line segment is said to be a *representation* of the free vector or is said to *represent* the free vector.
- **Identity Matrix** The $n\times n$ *identity matrix* is the matrix whose entry in row i and column i for $1\le i\le n$ is 1 and whose entries in row i and column j for $1\le i, j\le n$, and $i\ne j$ are all zero. The identity matrix is denoted by I.
- **Imaginary Axis** See *complex plane*.
- **Imaginary Number** An *imaginary number* is a complex number that can be expressed in the form bi where b is a real number.
- **Imaginary Part** See *complex number*.
- **Imaginary Unit** The *imaginary unit*, denoted by i, is the number corresponding to the point $(0,1)$ in the complex plane.
- **Incidence Matrix** The *incidence matrix of a network diagram* is the $n\times n$ matrix such that the entry in row i and column j is the number of edges that start at node i and end at node j.
- **Inverse Matrix** An $n\times n$ matrix A is *invertible* if there exists an $n\times n$ matrix B so that $\mathrm{AB}=\mathrm{BA}=\mathrm{I}$, where I is the $n\times n$ identity matrix. The matrix B, when it exists, is unique and is called the *inverse of* A and is denoted by A^{-1}.

- **Linear Function** A function $f:\mathbb{R}\to\mathbb{R}$ is called a *linear function* if it is a polynomial function of degree one, that is, a function with real number domain and range that can be put into the form $f(x)=mx+b$ for real numbers m and b. A linear function of the form $f(x)=mx+b$ is a linear transformation only if $b=0$.
- **Linear Transformation** A function $\mathrm{L}:\mathbb{R}^n\to\mathbb{R}^n$ for a positive integer n is a *linear transformation* if the following two properties hold:
  - $\mathrm{L}(\mathbf{x}+\mathbf{y})=\mathrm{L}(\mathbf{x})+\mathrm{L}(\mathbf{y})$ for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, and
  - $\mathrm{L}(k\mathbf{x})=k\cdot\mathrm{L}(\mathbf{x})$ for all $\mathbf{x}\in\mathbb{R}^n$ and $k\in\mathbb{R}$,

  where $\mathbf{x}\in\mathbb{R}^n$ means that $\mathbf{x}$ is a point in $\mathbb{R}^n$.
- **Linear Transformation Induced by Matrix A** Given a 2×2 matrix A, the *linear transformation induced by matrix* A is the linear transformation L given by the formula $\mathrm{L}\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\mathrm{A}\cdot\begin{bmatrix}x\\y\end{bmatrix}$. Given a 3×3 matrix A, the *linear transformation induced by matrix* A is the linear transformation L given by the formula $\mathrm{L}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\mathrm{A}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}$.
- **Matrix** An $m\times n$ *matrix* is an ordered list of nm real numbers, $a_{11},a_{12},\ldots,a_{1n}$, $a_{21},a_{22},\ldots,a_{2n},\ldots,a_{m1},a_{m2},\ldots,a_{mn}$, organized in a rectangular array of m rows and n columns: $\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}$. The number a_{ij} is called the *entry in row i and column j*.
- **Matrix Difference** Let A be an $m\times n$ matrix whose entry in row i and column j is a_{ij}, and let B be an $m\times n$ matrix whose entry in row i and column j is b_{ij}. Then, the *matrix difference* $\mathrm{A}-\mathrm{B}$ is the $m\times n$ matrix whose entry in row i and column j is $a_{ij}-b_{ij}$.
- **Matrix Product** Let A be an $m\times n$ matrix whose entry in row i and column j is a_{ij}, and let B be an $n\times p$ matrix whose entry in row i and column j is b_{ij}. Then, the *matrix product* AB is the $m\times p$ matrix whose entry in row i and column j is $a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$.
- **Matrix Scalar Multiplication** Let k be a real number, and let A be an $m\times n$ matrix whose entry in row i and column j is a_{ij}. Then, the *scalar product* $k\cdot\mathrm{A}$ is the $m\times n$ matrix whose entry in row i and column j is $k\cdot a_{ij}$.
- **Matrix Sum** Let A be an $m\times n$ matrix whose entry in row i and column j is a_{ij}, and let B be an $m\times n$ matrix whose entry in row i and column j is b_{ij}. Then, the *matrix sum* $\mathrm{A}+\mathrm{B}$ is the $m\times n$ matrix whose entry in row i and column j is $a_{ij}+b_{ij}$.
- **Modulus** The *modulus* of a complex number z, denoted $|z|$, is the distance from the origin to the point corresponding to z in the complex plane. If $z=a+bi$, then $|z|=\sqrt{a^2+b^2}$.
- **Network Diagram** A *network diagram* is a graphical representation of a directed graph where the n vertices are drawn as circles with each circle labeled by a number 1 through n and the directed edges are drawn as segments or arcs with the arrow pointing from the tail vertex to the head vertex.

- **Opposite Vector** For a vector $\vec{v}$ represented by the directed line segment $\overrightarrow{AB}$, the *opposite vector*, denoted $-\vec{v}$, is the vector represented by the directed line segment $\overrightarrow{BA}$. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $-\vec{v} = \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix}$.

- **Polar Form of a Complex Number** The *polar form of a complex number z* is $r(\cos(\theta) + i\sin(\theta))$ where $r = |z|$ and $\theta = \arg(z)$.

- **Position Vector** For a point $P(v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$, the *position vector* $\vec{v}$, denoted by $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ or $\langle v_1, v_2, \ldots, v_n \rangle$, is a free vector $\vec{v}$ that is represented by the directed line segment $\overrightarrow{OP}$ from the origin $O(0, 0, 0, \ldots, 0)$ to the point P. The real number v_i is called the ith *component* of the vector $\vec{v}$.

- **Real Coordinate Space** For a positive integer n, the n-dimensional *real coordinate space*, denoted $\mathbb{R}^n$, is the set of all n-tuple of real numbers equipped with a distance function d that satisfies

$$d\left[(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\right] = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

for any two points in the space. One-dimensional real coordinate space is called a *number line*, and the two-dimensional real coordinate space is called the *Cartesian plane*.

- **Rectangular Form of a Complex Number** The *rectangular form of a complex number z* is $a + bi$ where z corresponds to the point (a, b) in the complex plane and i is the imaginary unit. The number a is called the *real part* of $a + bi$, and the number b is called the *imaginary part* of $a + bi$.

- **Translation by a Vector in Real Coordinate Space** A *translation by a vector* $\vec{v}$ *in* $\mathbb{R}^n$ is the translation transformation $T_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}^n$ given by the map that takes $\vec{x} \mapsto \vec{x} + \vec{v}$ for all $\vec{x}$ in $\mathbb{R}^n$. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $T_{\vec{v}}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_1 + v_1 \\ x_2 + v_2 \\ \vdots \\ x_n + v_n \end{bmatrix}$ for all $\vec{x}$ in $\mathbb{R}^n$.

- **Vector Addition** For vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$, the sum $\vec{v} + \vec{w}$ is the vector whose ith component is the sum of the ith components of $\vec{v}$ and $\vec{w}$ for $1 \le i \le n$. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ in $\mathbb{R}^n$, then $\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$.

- **Vector Magnitude** The *magnitude* or *length* of a vector $\vec{v}$, denoted $|\vec{v}|$ or $\|\vec{v}\|$, is the length of any directed line segment that represents the vector. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$, which is the distance from the origin to the associated point $P(v_1, v_2, \ldots, v_n)$.
- **Vector Representation of a Complex Number** The *vector representation of a complex number* z is the position vector $\vec{z}$ associated to the point z in the complex plane. If $z = a + bi$ for two real numbers a and b, then $\vec{z} = \begin{bmatrix} a \\ b \end{bmatrix}$.
- **Vector Scalar Multiplication** For a vector $\vec{v}$ in $\mathbb{R}^n$ and a real number k, the scalar product $k \cdot \vec{v}$ is the vector whose ith component is the product of k and the ith component of $\vec{v}$ for $1 \le i \le n$. If k is a real number and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $k \cdot \vec{v} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}$.
- **Vector Subtraction** For vectors $\vec{v}$ and $\vec{w}$, the difference $\vec{v} - \vec{w}$ is the sum of $\vec{v}$ and the opposite of $\vec{w}$; that is, $\vec{v} - \vec{w} = \vec{v} + (-\vec{w})$. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ in $\mathbb{R}^n$, then $\vec{v} - \vec{w} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \\ \vdots \\ v_n - w_n \end{bmatrix}$.
- **Zero Matrix** The $m \times n$ *zero matrix* is the $m \times n$ matrix in which all entries are equal to zero. For example, the 2×2 zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and the 3×3 zero matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
- **Zero Vector** The *zero vector* in $\mathbb{R}^n$ is the vector in which each component is equal to zero. For example, the zero vector in $\mathbb{R}^2$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and the zero vector in $\mathbb{R}^3$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Module 2

- **Argument** The *argument* of the complex number z is the radian (or degree) measure of the counterclockwise rotation of the complex plane about the origin that maps the initial ray (i.e., the ray corresponding to the positive real axis) to the ray from the origin through the complex number z in the complex plane. The argument of z is denoted $\arg(z)$.
- **Bound Vector** A *bound vector* is a directed line segment (an *arrow*). For example, the directed line segment $\overrightarrow{AB}$ is a bound vector whose initial point (or *tail*) is A and terminal point (or *tip*) is B.

Bound vectors are *bound* to a particular location in space. A bound vector $\overline{AB}$ has a magnitude given by the length of $\overline{AB}$ and direction given by the ray $\overline{AB}$. Many times, only the magnitude and direction of a bound vector matters, not its position in space. In that case, any translation of that bound vector is considered to represent the same free vector.

- **Complex Number** A *complex number* is a number that can be represented by a point in the complex plane. A complex number can be expressed in two forms:

  1. The *rectangular form* of a complex number z is $a+bi$ where z corresponds to the point (a,b) in the complex plane, and i is the imaginary unit. The number a is called the *real part* of $a+bi$, and the number b is called the *imaginary part* of $a+bi$. Note that both the real and imaginary parts of a complex number are themselves real numbers.
  2. For $z \neq 0$, the *polar form* of a complex number z is $r(\cos(\theta)+i\sin(\theta))$ where $r=|z|$ and $\theta = \arg(z)$, and i is the imaginary unit.

- **Complex Plane** The *complex plane* is a Cartesian plane equipped with addition and multiplication operators defined on ordered pairs by the following:
  - Addition: $(a,b)+(c,d)=(a+c,b+d)$.
    When expressed in rectangular form, if $z=a+bi$ and $w=c+di$, then $z+w=(a+c)+(b+d)i$.
  - Multiplication: $(a,b)\cdot(c,d)=(ac-bd,ad+bc)$.
    When expressed in rectangular form, if $z=a+bi$ and $w=c+di$, then $z\cdot w=(ac-bd)+(ad+bc)i$. The horizontal axis corresponding to points of the form $(x,0)$ is called the *real axis*, and a vertical axis corresponding to points of the form $(0,y)$ is called the *imaginary axis*.

- **Conjugate** The *conjugate* of a complex number of the form $a+bi$ is $a-bi$. The conjugate of z is denoted $\overline{z}$.

- **Determinant of 2×2 Matrix** The *determinant of the* 2×2 *matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number computed by evaluating $ad-bc$ and is denoted by $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$.

- **Determinant of 3×3 Matrix** The *determinant of the* 3×3 *matrix* $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the number computed by evaluating the expression,

  $a_{11}\det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{12}\det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right) + a_{13}\det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right)$,

  and is denoted by $\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$.

- **Directed Graph** A *directed graph* is an ordered pair $D(V,E)$ with
  - V a set whose elements are called *vertices* or *nodes*, and
  - E a set of ordered pairs of vertices, called *arcs* or *directed edges*.

- **Directed Segment** A *directed segment* $\overline{\mathrm{AB}}$ is the line segment AB together with a direction given by connecting an initial point A to a terminal point B.
- **Free Vector** A *free vector* is the equivalence class of all directed line segments (arrows) that are equivalent to each other by translation. For example, scientists often use free vectors to describe physical quantities that have magnitude and direction only, freely placing an arrow with the given magnitude and direction anywhere in a diagram where it is needed. For any directed line segment in the equivalence class defining a free vector, the directed line segment is said to be a *representation* of the free vector or is said to *represent* the free vector.
- **Identity Matrix** The $n \times n$ *identity matrix* is the matrix whose entry in row i and column i for $1 \le i \le n$ is 1 and whose entries in row i and column j for $1 \le i, j \le n$, and $i \ne j$ are all zero. The identity matrix is denoted by I.
- **Imaginary Axis** See *complex plane*.
- **Imaginary Number** An *imaginary number* is a complex number that can be expressed in the form bi where b is a real number.
- **Imaginary Part** See *complex number*.
- **Imaginary Unit** The *imaginary unit*, denoted by i, is the number corresponding to the point $(0,1)$ in the complex plane.
- **Incidence Matrix** The *incidence matrix of a network diagram* is the $n \times n$ matrix such that the entry in row i and column j is the number of edges that start at node i and end at node j.
- **Inverse Matrix** An $n \times n$ matrix A is *invertible* if there exists an $n \times n$ matrix B so that $\mathrm{AB} = \mathrm{BA} = \mathrm{I}$, where I is the $n \times n$ identity matrix. The matrix B, when it exists, is unique and is called the *inverse of* A and is denoted by A^{-1}.
- **Linear Function** A function $f : \mathbb{R} \to \mathbb{R}$ is called a *linear function* if it is a polynomial function of degree one, that is, a function with real number domain and range that can be put into the form $f(x) = mx + b$ for real numbers m and b. A linear function of the form $f(x) = mx + b$ is a linear transformation only if $b = 0$.
- **Linear Transformation** A function $\mathrm{L} : \mathbb{R}^n \to \mathbb{R}^n$ for a positive integer n is a *linear transformation* if the following two properties hold:
    - $\mathrm{L}(\mathbf{x} + \mathbf{y}) = \mathrm{L}(\mathbf{x}) + \mathrm{L}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and
    - $\mathrm{L}(k\mathbf{x}) = k \cdot \mathrm{L}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $k \in \mathbb{R}$,

  where $\mathbf{x} \in \mathbb{R}^n$ means that $\mathbf{x}$ is a point in $\mathbb{R}^n$.
- **Linear Transformation Induced by Matrix A** Given a 2×2 matrix A, the *linear transformation induced by matrix* A is the linear transformation L given by the formula $\mathrm{L}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \mathrm{A} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$. Given a 3×3 matrix A, the *linear transformation induced by matrix* A is the linear transformation L given by the formula $\mathrm{L}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \mathrm{A} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

- **Matrix** An $m \times n$ *matrix* is an ordered list of nm real numbers, $a_{11}, a_{12}, \ldots,$ $a_{1n}, a_{21}, a_{22}, \ldots, a_{2n}, \ldots, a_{m1}, a_{m2}, \ldots, a_{mn}$, organized in a rectangular array of m rows and n columns: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$. The number a_{ij} is called the *entry in row i and column j.*
- **Matrix Difference** Let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij}, and let B be an $m \times n$ matrix whose entry in row i and column j is b_{ij}. Then, the *matrix difference* $A - B$ is the $m \times n$ matrix whose entry in row i and column j is $a_{ij} - b_{ij}$.
- **Matrix Product** Let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij}, and let B be an $n \times p$ matrix whose entry in row i and column j is b_{ij}. Then, the *matrix product* AB is the $m \times p$ matrix whose entry in row i and column j is $a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.
- **Matrix Scalar Multiplication** Let k be a real number, and let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij}. Then, the *scalar product* $k \cdot A$ is the $m \times n$ matrix whose entry in row i and column j is $k \cdot a_{ij}$.
- **Matrix Sum** Let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij}, and let B be an $m \times n$ matrix whose entry in row i and column j is b_{ij}. Then, the *matrix sum* $A + B$ is the $m \times n$ matrix whose entry in row i and column j is $a_{ij} + b_{ij}$.
- **Modulus** The *modulus* of a complex number z, denoted $|z|$, is the distance from the origin to the point corresponding to z in the complex plane. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.
- **Network Diagram** A *network diagram* is a graphical representation of a directed graph where the n vertices are drawn as circles with each circle labeled by a number 1 through n and the directed edges are drawn as segments or arcs with the arrow pointing from the tail vertex to the head vertex.
- **Opposite Vector** For a vector $\vec{v}$ represented by the directed line segment $\overrightarrow{AB}$, the *opposite vector*, denoted $-\vec{v}$, is the vector represented by the directed line segment $\overrightarrow{BA}$. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $-\vec{v} = \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix}$.
- **Polar Form of a Complex Number** The *polar form of a complex number z* is $r(\cos(\theta) + i\sin(\theta))$ where $r = |z|$ and $\theta = \arg(z)$.
- **Position Vector** For a point $P(v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$, the *position vector* $\vec{v}$, denoted by $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ or $\langle v_1, v_2, \ldots, v_n \rangle$, is a free vector $\vec{v}$ that is represented by the directed line segment $\overrightarrow{OP}$ from the origin $O(0, 0, 0, \ldots, 0)$ to the point P. The real number v_i is called the *ith component* of the vector $\vec{v}$.

- **Real Coordinate Space** For a positive integer n, the n-dimensional *real coordinate space*, denoted R^n, is the set of all n-tuple of real numbers equipped with a distance function d that satisfies

$$d\left[(x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n)\right]=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+\cdots+(y_n-x_n)^2}$$

for any two points in the space. One-dimensional real coordinate space is called a *number line*, and the two-dimensional real coordinate space is called the *Cartesian plane*.

- **Rectangular Form of a Complex Number** The *rectangular form of a complex number z* is $a+bi$ where z corresponds to the point (a,b) in the complex plane and i is the imaginary unit. The number a is called the *real part* of $a+bi$, and the number b is called the *imaginary part* of $a+bi$.

- **Translation by a Vector in Real Coordinate Space** A *translation by a vector $\vec{v}$ in $\mathbb{R}^n$* is the translation transformation $T_{\vec{v}}:\mathbb{R}^n\to\mathbb{R}^n$ given by the map that takes $\vec{x}\mapsto\vec{x}+\vec{v}$ for all $\vec{x}$ in R^n. If $\vec{v}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$ in $\mathbb{R}^n$, then $T_{\vec{v}}\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right)=\begin{bmatrix}x_1+v_1\\x_2+v_2\\\vdots\\x_n+v_n\end{bmatrix}$ for all $\vec{x}$ in $\mathbb{R}^n$.

- **Vector** A *vector* is described as either a bound or free vector depending on the context. We refer to both bound and free vectors as vectors throughout this module.

- **Vector Addition** For vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$, the sum $\vec{v}+\vec{w}$ is the vector whose ith component is the sum of the ith components of $\vec{v}$ and $\vec{w}$ for $1\le i\le n$. If $\vec{v}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$ and $\vec{w}=\begin{bmatrix}w_1\\w_2\\\vdots\\w_n\end{bmatrix}$ in $\mathbb{R}^n$, then $\vec{v}+\vec{w}=\begin{bmatrix}v_1+w_1\\v_2+w_2\\\vdots\\v_n+w_n\end{bmatrix}$.

- **Vector Magnitude** The *magnitude* or *length* of a vector $\vec{v}$, denoted $|\vec{v}|$ or $\vec{v}$, is the length of any directed line segment that represents the vector. If $\vec{v}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$ in $\mathbb{R}^n$, then $|\vec{v}|=\sqrt{v_1^2+v_2^2+\cdots+v_n^2}$, which is the distance from the origin to the associated point $P(v_1, v_2, \ldots, v_n)$.

- **Vector Representation of a Complex Number** The *vector representation of a complex number z* is the position vector $\vec{z}$ associated to the point z in the complex plane. If $z=a+bi$ for two real numbers a and b, then $\vec{z}=\begin{bmatrix}a\\b\end{bmatrix}$.

- **Vector Scalar Multiplication** For a vector $\vec{v}$ in $\mathbb{R}^n$ and a real number k, the scalar product $k \cdot \vec{v}$ is the vector whose ith component is the product of k and the ith component of $\vec{v}$ for $1 \le i \le n$. If k is a real number and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $k \cdot \vec{v} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}$.
- **Vector Subtraction** For vectors $\vec{v}$ and $\vec{w}$, the difference $\vec{v} - \vec{w}$ is the sum of $\vec{v}$ and the opposite of $\vec{w}$; that is, $\vec{v} - \vec{w} = \vec{v} + (-\vec{w})$. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ in $\mathbb{R}^n$, then $\vec{v} - \vec{w} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \\ \vdots \\ v_n - w_n \end{bmatrix}$.
- **Zero Matrix** The $m \times n$ *zero matrix* is the $m \times n$ matrix in which all entries are equal to zero. For example, the 2×2 zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and the 3×3 zero matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
- **Zero Vector** The *zero vector* in $\mathbb{R}^n$ is the vector in which each component is equal to zero. For example, the zero vector in $\mathbb{R}^2$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and the zero vector in $\mathbb{R}^3$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Module 3

- **Ellipse** An *ellipse* is the set of all points in a plane such that the sum of the distances from two points (foci) to any point on the line is constant. Given k, foci A and B, and any point P on the ellipse, $\mathrm{PA} + \mathrm{PB} = k$.

  Standard equation for an ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

  An ellipse whose foci points are the same point, that is, $\mathrm{A} = \mathrm{B}$, is a circle.
- **Horizontal Asymptote** Let L be a real number. The line given by the equation $y = \mathrm{L}$ is a *horizontal asymptote* of the graph of $y = f(x)$ if at least one of the following statements is true.
  - As x approaches infinity, $f(x)$ approaches L.
  - As x approaches negative infinity, $f(x)$ approaches L.
- **Hyperbola** A *hyperbola* is the set of points in a plane whose distances to two fixed points A and B, called the foci, have a constant difference. Given P and a positive constant, k, $|\mathrm{PA} - \mathrm{PB}| = k$.

  Standard equation for a hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
- **Vertical Asymptote** Let a be a real number. The line given by the equation $x = a$ is a *vertical asymptote* of the graph of $y = f(x)$ if at least one of the following statements is true.
  - As x approaches a, $f(x)$ approaches infinity.
  - As x approaches a, $f(x)$ approaches negative infinity.

Module 4

No New or Recently Introduced Terms

Module 5

- **Combination of *k* Items Selected from a Set of *n* Distinct Items** An unordered set of k items selected from a set of n distinct items.
- **Continuous Random Variables** A random variable for which the possible values form an entire interval along the number line.
- **Discrete Random Variables** A random value for which the possible values are isolated points along the number line.
- **Empirical Probability** A probability that has been estimated by observing a large number of outcomes of a chance experiment or values of a random variable.
- **Expected Value of a Random Variable** The long-run average value expected over a large number of observations of the value of a random variable.
- **Fundamental Counting Principle** Let n_1 be the number of ways the first step or event can occur and n_2 be the number of ways the second step or event can occur. Continuing in this way, let n_k be the number of ways the kth stage or event can occur. Then, based on the fundamental counting principle, the total number of different ways the process can occur is $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$.
- **General Multiplication Rule** A probability rule for calculating the probability of the intersection of two events.
- **Long-Run Behavior of a Random Variable** The behavior of the random variable over a very long sequence of observations.
- **Permutation of *k* Items Selected from a Set of *n* Distinct Items** An ordered sequence of k items selected from a set of n distinct items.
- **Probability Distribution** A table or graph that provides information about the long-run behavior of a random variable.
- **Probability Distribution of a Discrete Random Variable** A table or graph that specifies the possible values of the random variable and the associated probabilities.
- **Random Variable** A variable whose possible values are based on the outcome of a random event.
- **Theoretical Probability** A probability calculated by assigning a probability to all possible outcomes in the sample space for a chance experiment.
- **Uniform Probability Model** A probability distribution that assigns equal probability to each possible outcome of a chance experiment.

Notes

CHAPTER 2

1. achievethecore.org. "The Common Core State Standards Shifts in Mathematics." http://achievethecore.org/shifts-mathematics (accessed Nov. 24, 2015).
2. corestandards.org. "High School Publishers' Criteria for the Common Core State Standards for Mathematics." achievethecore.org (accessed Feb. 4, 2016).
3. Ibid.
4. achievethecore.org. "The Common Core Standards Shifts in Mathematics."
5. Ibid.
6. Ibid.
7. Ibid.
8. Ibid.

CHAPTER 3

1. This standard will be assessed in Algebra II by ensuring that some modeling tasks (involving Algebra II content or securely held content from previous grades and courses) require the student to create a quantity of interest in the situation being described (i.e., this is not provided in the task). For example, in a situation involving periodic phenomena, the student might autonomously decide that amplitude is a key variable in a situation and then choose to work with peak amplitude.
2. In Algebra II, tasks are limited to polynomial, rational, or exponential expressions. Examples: see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. In the equation $x^2 + 2x + 1 + y^2 = 9$, see an opportunity to rewrite the first three terms as $(x+1)^2$, thus recognizing the equation of a circle with radius 3 and center $(-1, 0)$. See $\frac{(x^2+4)}{(x^2+3)}$ as $\frac{((x^2+3)+1)}{(x^2+3)}$, thus recognizing an opportunity to write it as $1 + \left(\frac{1}{(x^2+3)}\right)$. Tasks include the sum or difference of cubes (in one variable) and factoring by grouping.
3. Tasks include problems that involve interpreting the remainder theorem from graphs and in problems that require long division.
4. In Algebra II, tasks include quadratic, cubic, and quartic polynomials and polynomials for which factors are not provided. For example, find the zeros of $(x^2 - 1)(x^2 + 1)$.
5. Prove *and apply*.

6. Tasks include rewriting rational expressions that are in the form of a complex fraction.
7. In Algebra II, tasks are limited to simple rational or radical equations.
8. In Algebra II, in the case of equations having roots with nonzero imaginary parts, students write the solutions as $a \pm bi$, where a and b are real numbers.
9. In Algebra II, tasks are limited to 3×3 systems.
10. According to the CCSSM (p. 57), "Modeling is best interpreted not as a collection of isolated topics but in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★). The star symbol sometimes appears on the heading for a group of standards; in that case, it should be understood to apply to all standards in that group."
11. Also explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to their reciprocal functions.
12. Choose trigonometric functions to model periodic phenomena with specified phase shift.
13. Tasks have a real-world context. In Algebra II, tasks include exponential functions with domains not in the integers and trigonometric functions.
14. This standard includes expressions where either base or exponent may contain variables.
15. This standard will be assessed in Algebra II by ensuring that some modeling tasks (involving Algebra II content or securely held content from previous grades and courses) require the student to create a quantity of interest in the situation being described (i.e., this is not provided in the task). For example, in a situation involving periodic phenomena, the student might autonomously decide that amplitude is a key variable in a situation and then choose to work with peak amplitude.
16. Tasks have a real-world context. As described in the standard, there is an interplay between the mathematical structure of the expression and the structure of the situation, such that choosing and producing an equivalent form of the expression reveal something about the situation. In Algebra II, tasks include exponential expressions with rational or real exponents.
17. This standard includes using the summation notation symbol.
18. Tasks have a real-world context. In Algebra II, tasks include exponential equations with rational or real exponents, rational functions, and absolute value functions.
19. In Algebra II, tasks may involve any of the function types mentioned in the standard.
20. This standard is Supporting Content in Algebra II. This standard should support the Major Content in F-BF.A.2 for coherence.
21. Tasks have a real-world context. In Algebra II, tasks may involve polynomial, exponential, logarithmic, and trigonometric functions.
22. Tasks have a real-world context. In Algebra II, tasks may involve polynomial, exponential, logarithmic, and trigonometric functions.
23. Tasks include knowing and applying $A = Pe^{rt}$ and $A = P(1 + \frac{r}{n})^{nt}$.
24. In Algebra II, tasks may involve polynomial, exponential, logarithmic, and trigonometric functions.
25. Tasks have a real-world context. In Algebra II, tasks may involve linear functions, quadratic functions, and exponential functions.

26. Combining functions also includes composition of functions.
27. In Algebra II, tasks may involve polynomial, exponential, logarithmic, and trigonometric functions. Tasks may involve recognizing even and odd functions.
28. In Algebra II, tasks will include solving multi-step problems by constructing linear and exponential functions.
29. Students learn terminology that logarithm without a base specified is base 10 and that natural logarithm always refers to base e.
30. Tasks have a real-world context. In Algebra II, tasks include exponential functions with domains not in the integers.

CHAPTER 6

1. This standard is assessed in Algebra II by ensuring that some modeling tasks (involving Algebra II content or securely held content from previous grades and courses) require the student to create a quantity of interest in the situation being described (i.e., this is not provided in the task). For example, in a situation involving periodic phenomena, the student might autonomously decide that amplitude is a key variable in a situation and then choose to work with peak amplitude.
2. In Algebra II, tasks are limited to polynomial, rational, or exponential expressions. Examples: see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. In the equation $x^2 + 2x + 1 + y^2 = 9$, see an opportunity to rewrite the first three terms as $(x+1)^2$, thus recognizing the equation of a circle with radius 3 and center $(-1, 0)$. See $\frac{(x^2+4)}{(x^2+3)}$ as $\frac{((x^2+3)+1)}{(x^2+3)}$, thus recognizing an opportunity to write it as $1 + \left(\frac{1}{(x^2+3)}\right)$.
3. Tasks include problems that involve interpreting the remainder theorem from graphs and in problems that require long division.
4. In Algebra II, tasks include quadratic, cubic, and quartic polynomials and polynomials for which factors are not provided. For example, find the zeros of $(x^2 - 1)(x^2 + 1)$.
5. Tasks include rewriting rational expressions that are in the form of a complex fraction.
6. In Algebra II, tasks are limited to simple rational or radical equations.
7. In Algebra II, in the case of equations having roots with nonzero imaginary parts, students write the solutions as $a \pm bi$, where a and b are real numbers.
8. In Algebra II, tasks are limited to 3×3 systems.
9. This standard includes expressions where either base or exponent may contain variables.
10. This standard is assessed in Algebra II by ensuring that some modeling tasks (involving Algebra II content or securely held content from previous grades and courses) require students to create a quantity of interest in the situation being described (i.e., this is not provided in the task). For example, in a situation involving periodic phenomena, students might autonomously decide that amplitude is a key variable in a situation and then choose to work with peak amplitude.

11. Tasks have a real-world context. As described in the standard, there is an interplay between the mathematical structure of the expression and the structure of the situation, such that choosing and producing an equivalent form of the expression reveal something about the situation. In Algebra II, tasks include exponential expressions with rational or real exponents.
12. This standard includes using the summation notation symbol.
13. Tasks have a real-world context. In Algebra II, tasks include exponential equations with rational or real exponents, rational functions, and absolute value functions.
14. In Algebra II, tasks may involve any of the function types mentioned in the standard.
15. This standard is Supporting Content in Algebra II. This standard should support the Major Content in F-BF.2 for coherence.
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20. Tasks have a real-world context. In Algebra II, tasks may involve linear functions, quadratic functions, and exponential functions.
21. Combining functions also includes composition of functions.
22. In Algebra II, tasks may involve polynomial, exponential, logarithmic, and trigonometric functions. Tasks may involve recognizing even and odd functions.
23. In Algebra II, tasks include solving multi-step problems by constructing linear and exponential functions.
24. Students learn terminology that logarithm without a base specified is base 10 and that natural logarithm always refers to base e.
25. Tasks have a real-world context. In Algebra II, tasks include exponential functions with domains not in the integers.

CHAPTER 7

1. *Matched* can be replaced with *assigned* after students understand that each element of X is matched to exactly one element of Y.

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